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NEW SERIES

OF THE

MATHEMATICAL

REPOSITORY.

By **THOMAS LEYBOURN**,
Of the Royal Military College.

VOL. II.

London :

**PRINTED AND SOLD BY W. GLENDINNING, NO. 25,
HATTON GARDEN.**

1809.

Sci 890.10

(1927)

207

1927

1927

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II. The first, second, and third parts of the vol. must follow in their order. The two sets of questions of vol. 3. (viz. the signatures A and B) which are stiched up with the numbers forming the second vol. must be preserved apart till the third vol. be completed.

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NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

I. MATHEMATICAL QUESTIONS,

To be Answered in Number VI.

I. QUESTION 121, by SCOTICUS.

If α and β denote given arches, and
 $\sec. \alpha \sec. \beta + \tan. \alpha \tan. \phi = \sec. \beta$.

Required the values of $\sec. \phi$ and $\tan. \phi$?

II. QUESTION 122, by AMICUS.

If two tangents PE, PF, be drawn to a conic section, and the line EF joining the points of contact be produced to meet the directrix in G; then if PI be drawn through the focus, to meet EF in I, EI will be to IF as EG to FG, required the demonstration?

III. QUESTION 123, by Mr. JAMES WHALLEY, Bolton.

To construct a trapezium there is given one of the sides, both the diagonals and the angle formed by their intersection, and the angle made by either diagonal and side opposite to the given one?

VOL. II.

A

IV.

IV. QUESTION 124, *by* SENEX.

Let there be two given points in the circumference of a circle given in magnitude and position: a circle may be found, given also in magnitude and position, such, that if from any point in its circumference straight lines be drawn to the given points, and a tangent to the given circle; the square of the tangent shall have to the rectangle of the lines drawn to the given points a given ratio?

V. QUESTION 125, *by* Mr. JOSEPH BREWER, *Preslon*.

Two right lines being given by position, a circle may be found, such, that if another circle be described upon any radius thereof as a diameter, the chord of the arc of this latter circle intercepted by the lines given by position shall be of a given length?

VI. QUESTION 126, *by* Mr. J. JOHNSON.

Given the position of the centres of gravity of the three triangles, formed by lines drawn from the angular points of a triangle to the centre of the inscribed circle; to determine the position of the triangle?

VII. QUESTION 127, *by* HYPATIA.

Suppose $u = \frac{a^{a-n} - x^{a-n}}{a - x}$, what is the value of u when $= x a$?

VIII. QUESTION 128, *by* URSA MINOR.

Let AB, AC, be two straight lines given by position, and let a straight line DE meet them in D and E so that AD + AE is equal to a given line, and let DE be divided at V, so that DV may have to VE a given ratio. Required the locus of the point V?

IX.

IX. QUESTION 129, *by Mr. CUNLIFFE, R. M. College.*

To find two rational numbers, such, that if a given number n be subtracted from either of them or from their sum or difference, the four remainders shall all be rational squares ?

X. QUESTION 130, *by TRIGONOMETRICUS.*

It is required to demonstrate that the tangent of an arch of 9° is equal to $1 + \sqrt{5} - \sqrt{5 + 2\sqrt{5}}$, the radius being 1 ?

The next Six Questions are selected from the Cambridge University Calendars for 1804 and 1805.

XI. QUESTION 131, *from the CAMB. UN. CAL.*

A given parabolic conoid is put into a fluid, and when it is in a position of permanent equilibrium the base is wholly extant, and inclined to the horizon; having given its specific gravity relatively to that of the fluid, it is required to determine the dimensions of the part immersed ?

XII. QUESTION 132, *from the CAMB. UN. CAL.*

Find the centre of gravity of a cylindrical portion of the atmosphere measured from the earth's surface, the force of gravity being constant ?

XIII. QUESTION 133, *from the CAMB. UN. CAL.*

Required the nature of the curve along which a heavy body, descending by the force of gravity, will press upon the curve at any point, with a force proportional to the ordinate at that point ?

XIV. QUESTION 134, *from the CAMB. UN. CAL.*

Let the curve APp be generated by taking the ordinate MP always equal to the difference of the chord AC , and the versed sine AM , of the circular arc AC . It is required to determine the greatest ordinate of the curve, and also its area?

XV. QUESTION 135, *from the CAMB. UN. CAL.*

Determine the apparent magnitude of a straight rectilinear object, placed at a given depth, parallel to a surface of water; the eye being situated at any point in the plane passing through the object perpendicular to the surface?

XVI. QUESTION 136, *from the CAMB. UN. CAL.*

Let the roots of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, be $a, b, c, \&c.$ and those of the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} - \&c. = 0$, be $\alpha, \beta, \gamma, \&c.$; then, if when $a, b, c, \&c.$ are successively substituted in the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, the results are $P, Q, R, \&c.$ and when $\alpha, \beta, \gamma, \&c.$ are substituted in the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} - \&c. = 0$, the results are $p, q, r, \&c.$ it will be as $P \times Q \times R, \&c. : p \times q \times r, \&c. :: 1 : n^n$. Required the demonstration?

XVII. QUESTION 137, *by Mr. HENRY LIGHTBOWN.*

Suppose a heavy flexible chain of a given length having its ends fastened together to be thrown over the vertex of a given square pyramid whose surface is perfectly polished; what portion of the surface will be included between the vertex and the chain when it has descended as low as it can get?

XVIII. QUESTION 138, by Mr. THOS. BAZLEY, Bolton.

Suppose a perfectly flexible chain of a given length to have one end fastened to an immoveable tack, and a ring at the other end to be put upon a perfectly smooth inflexible line or rod given by position; it is required to determine that point upon the rod at which the ring will rest?

XIX. QUESTION 139, by Mr. LOWRY.

In a given trapezium to inscribe another, such that three of its sides may pass through given points, and the remaining side be of a given length?

XX. QUESTION 140, by Mr. W. SIMPSON, Bolton.

$$\begin{aligned}
 & n \times \left\{ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c. \text{ ad infinitum} \right\} \\
 &= \frac{x^n-1}{x^n} + \frac{1}{2} \left(\frac{x^n-1}{x^n} \right)^2 + \frac{1}{3} \left(\frac{x^n-1}{x^n} \right)^3 + \frac{1}{4} \left(\frac{x^n-1}{x^n} \right)^4 + \frac{1}{5} \left(\frac{x^n-1}{x^n} \right)^5 \\
 &+ \&c. \text{ ad infinitum. } \text{ Required the demonstration?}
 \end{aligned}$$

XXI. QUESTION 141, by Mr. CUNLIFFE.

It is required to find three whole numbers such, that the sum of every two may be a cube; and also the sum of all the three a cube?

XXII. QUESTION 142, by Mr. CUNLIFFE.

P is a given point in the diameter AB produced of a given circle, let any line PCD be drawn cutting the circle in C and D, and let AC, BD, be drawn intersecting in I: now, if in BD produced there is continually taken DR = DI, required the equation and quadrature of the curve which is the locus of R?

XXIII.

XXIII. QUESTION 143, *by Mr. BAZLEY.*

Suppose a thread of a given length to be fastened at one end to a given point on a perfectly smooth horizontal plane having a straight edge; and a weight attached to the other end of the thread and brought to the edge of the horizontal plane with the thread stretched; then if the weight be suffered to descend by the force of gravity from this position; it is required to determine the time in which the thread will acquire any other possible assigned position upon the plane, together with the velocity of the descending weight in such position; also required the nature of the curve described by the descending weight?

XXIV. QUESTION 144, *by Mr. CUNLIFFE.*

Having given a triangle, it is required to determine a circle and concentric ellipse, together with the position of a right line passing through the said centre, such, that one of the angles of the triangle being always placed upon the right line, the other two angles shall be one in the circumference of the circle, and the other in the periphery of the ellipse?

XXV. QUESTION 145, *by Mr. I. I.*

If from a point without an ellipse or an hyperbola, two tangents be drawn to the curve, and likewise two straight lines be drawn to the foci, the angle contained by one tangent and the line drawn to one focus will be equal to the angle contained by the other tangent and the line drawn to the other focus?

XXVI. QUESTION 146, *by Mr. LOWRY.*

Let there be any number of straight lines AP, BP, CP, DP, &c. given by position, and let Z be a given point without them; it is required to describe a circle through the points P and Z, to intersect the given lines in A, B, C, D, &c. so that the sum of all the intercepted parts AP, BP, CP, DP, &c. may be equal to a given line?

XXVII.

XXVII. QUESTION 147, *by Mr. WM. WALLACE,*
R. M. College.

If x denote an arch of the meridian, and y denote the meridional parts corresponding to the latitude x , then

$$y = x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{725678} + \&c.$$

and by reverting the series

$$x = y - \frac{y^3}{6} + \frac{y^5}{24} - \frac{61y^7}{5040} + \frac{277y^9}{725678} - \&c.$$

where it is remarkable that the numeral coefficients of the powers of x are the same as those of the like powers of y , and the signs of the terms of the one series are all +, but those in the other series + and - alternately. Shew the reason of these peculiarities?

XXVIII. QUESTION 148, *by Mr. I. R.*

Let two circles, one of which is included within the other, be given in a plane, and let there be given a point, situated within both the circles, in the line joining the two centres; it is required to describe a circle, from the given point as a centre, to meet the peripheries of the given circles, so that the angle contained by two straight lines drawn from the given point to the two points of concurrence may be the least possible?

✓ XXIX. QUESTION 149, *by MECHANICUS.*

To place a straight rod, or beam, loaded with any given weights, so that it may be in equilibrio, resting on an upright prop, and one end touching a smooth vertical wall given by position?

XXX. PRIZE QUESTION 150, *by LIMENUS.*

Let A and B be two given points without a circle given in magnitude and position; a point C may be found within the circle such that if any straight line be drawn through it cutting the circle in F and G, and AF, BG, be joined, the square of AF will have to the square of BG the ratio which is compounded of the ratio of FC to CG, and a certain given ratio, and which ratio is also to be found?

Solutions to these Questions must come to hand (post-paid) by the first of February, 1806.



NEW SERIES OF THE MATHEMATICAL REPOSITORY.

II. MATHEMATICAL QUESTIONS

To be Answered in Number VII.

I. QUESTION 151, by PAPPUS.

Given the length of a circular arc, to find the length of the chord connecting its extremities, so that the area of the included segment may be the greatest possible?

II. QUESTION 152, by HERMODORUS.

To inscribe seven equal circles in a given circle?

III. QUESTION 153, by CENTROBARICOS.

The radius of the greater end of the frustum of a paraboloid being R , that of its less end r , and the height of the frustum h ; it is required to give a neat expression for the position of its centre of gravity?

IV. QUESTION 154, by DIOPHANTUS.

Find a trapezium which can be inscribed in a circle such, that its sides and area may be expressed by whole numbers?

V. QUESTION 155, by Mr. J. JOHNSON.

In a given spherical triangle, it is required to inscribe another, so that its perimeter may be the least possible?

VI. QUESTION 156, by LIMENUS.

If a straight line be drawn to meet the four sides of a trapezium in a, b, c, d , and the two diagonals in m and n ; then will ab be to cd as $am \times nb$ to $cn \times nd$. Required the demonstration?

VII. QUESTION 157, by J. W. LEEDS.

Let A be a given point in the periphery of a given circle, let any chord AC be drawn, and divided in the point E , so that the rectangle $AC \times CE$ shall be equal to a given space; it is required to determine the equation and quadrature of the locus of the point E ?

VIII. QUESTION 158, by JUVENIS.

Given $2xx - 3yx = 10y - 5xy$, to find the equation of the fluents?

IX. QUESTION 159, by A. M.

If from the extremities A and B of the diameter AB of a given semicircle ABC , two chords, AC, BD , be drawn intersecting each other in P , and BD bisecting the arc AC in D ; it is required to determine the equation and quadrature of the curve which is the locus of the point P ?

X. QUESTION 160, by Mr. CUNLIFFE, R. M. College.

Required the sum of the infinite series

$$1 - \frac{1}{2} \times (\frac{1}{2} + \frac{1}{2} \times (\frac{1}{2} + \frac{1}{2}) - \frac{1}{4} \times (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) + \frac{1}{8} \times (\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \&c.$$

VOL. II.

B

XI.

XI. QUESTION 161, *by Mr. W. SIMPSON, Bolton.*

Let ACB be a given triangle, D, L, Q given points in the side AB; through D draw any right line DEF, cutting AC, BC, in E and F respectively, and through E and F draw right lines LEP, QFP intersecting in P. Required the locus of P?

XII. QUESTION 162, *by AMICUS.*

If a given weight be sustained by means of a string whose ends are connected (by loops or rings) to two rods placed in the same vertical plane, and making given angles with the horizon; it is required to ascertain the position of the said weight when it rests in equilibrio, supposing it has liberty to slide freely along the string?

XIII. QUESTION 163, *by Mr. CUNLIFFE.*

Find such positive values of x , y , and z , in whole numbers, as will make the formulæ

$x^2 + xy + y^2$, $x^2 + xz + z^2$ and $y^2 + yz + z^2$,
all rational squares?

XIV. QUESTION 164, *by URSA MINOR.*

A straight line drawn through the focus of any conic section, to the point in which a diameter meets the directrix, will be perpendicular to a tangent at either extremity of that diameter?

XV. QUESTION 165, *by AMICUS.*

If one end of a given beam rest against a smooth vertical plane, and the other end be sustained by a weight fastened to a string, which passes over a pulley placed at a given point; it is required to determine the position of the beam when it rests in equilibrio?

XVI. QUESTION 166, *by X. Y.*

Suppose two weights connected by a string, which passes over a pulley, and that one of the weights is placed upon an inclined plane, while the other hangs in the air by the connecting string. It is required to determine the *locus* of the centre of gravity of the weights, supposing the relative position of the pulley and plane, the length of the string, and the weights to be all given?

XVII. QUESTION 167, *by LIMENUS.*

When any number of circles pass through the same two points, tangents drawn from any point in either of the two external circumferences to the remaining circles shall be to one another in a given ratio?

XVIII QUESTION 168, *by LAPUTIENSIS.*

Suppose two bodies whose weights are known to be connected by a flexible line or string passing through a small hole in a perfectly smooth horizontal plane, and let the body which is above the plane be projected thereon, with a given velocity at a given distance from the hole, in a direction perpendicular to the string which

which is kept tight by the other body hanging freely ; it is required to determine the velocity, time, and angle formed by the string with the first position, when the projected body has attained any other possible assigned distance from the hole through which the string passes ?

XIX. QUESTION 169, by HERMODORUS.

Given the vanishing line, its centre and distance, to find the perspective representation of a circle from the given representation of one radius, or of an inscribed triangle ?

XX. QUESTION 170, by HERMODORUS.

Given in position a circle and two points ; if from the two points be inflected two straight lines to any point of the circle, and from the other two points in which these lines meet the circle, there be inflected two other straight lines to a point in the circle, so that one of them may be parallel to a straight line given by position, then the other straight line will either tend to a given point, or make a given angle with a straight line tending to a given point ?

XXI. QUESTION 171, by ANALYTICUS.

Find a series for the n th power of the logarithm of a number ?

XXII. QUESTION 172, by SENEX.

It is required to find the nature or equation of a curve such, that any right line RPQ being drawn through a given point P , to cut the curve in two points R and Q , and tangents from these points produced to intersect each other in I , the angle QIR may always be of the same constant magnitude, that is always equal to a given angle ?

XXIII. QUESTION 173, by Mr. JOHN FLETCHER.

If the magnitude of a circle is given ; it is required to find the position of a right line RS , and that of a point P in another line given in position, from whence if PAB be drawn to cut the circle in A and B , and perpendiculars AR and BS are drawn to RS , their rectangle will be always a given magnitude ?

XXIV. QUESTION 174, by AMICUS.

If a given cone, suspended by its vertex, be made to revolve with its axis in a vertical plane ; it is required to determine the force exerted on the centre of suspension at any given position of the revolving body ?

XXV. QUESTION 175, by Mr. WALLACE, R. M. College.

Suppose a quadrilateral figure to be inscribed in another quadrilateral figure, agreeable to the conditions required in question 120 of the New Series of the Repository, then the opposite sides of the inscribed figure when produced, will intersect each other in the diagonals produced of the circumscribed figure. Required the demonstration ?

XXVI.

XXVI. QUESTION 176, by MARLOVIENSIS.

It is well known that the sum of the series

$$x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2} - \&c. \text{ ad infinitum, is ex-}$$

pressed by the fluent of $(x^{-1} \times \text{fl. } \frac{x}{1+x})$, or area of a trans-

cendent hyperbola. Under what circumstances, can the same sum be exhibited by the multiples or powers of the arc of a circle, and how is this to be investigated?

XXVII. QUESTION 177, by MARLOVIENSIS.

It is required to shew a simple practical method (capable of demonstration), of putting a globe or sphere into perspective; having given the magnitude and situation of the sphere, the height of the eye above the ground plane, and its distance from the plane of projection, which is, as usual, supposed to be perpendicular to the horizon, also to shew what situation the globe must have, that the contour of its perspective may be any proposed conic section?

XXVIII. QUESTION 178, by MECHANICUS.

If two given bodies A and B, connected by a string, be supported on an inclined plane, by means of the string's passing through a ring at A, and any given force be impressed on the body A, in a horizontal direction; it is required to determine the nature of the curve described by the body, and the circumstances of the motion?

XXIX. QUESTION 179, by Mr. WALLACE, R. M. College.

Suppose two particles of matter p and q , to be connected by a string pAq , and to rest in equilibrio upon two inclined planes AB, AC, the string being supposed to pass over the common section of the planes at the point A; then, if a circle be described through the points p , A, q , and the particles without changing their position are disengaged from the string pAq , and connected by another paq , which passes over a peg fastened at any point a in the circumference of the circle, the particles p and q will still rest in equilibrio upon the planes. Required the demonstration?

XXX. PRIZE QUESTION 180, by GEOMETRICUS.

If the opposite sides of a hexagon, inscribed in a circle, be produced till they meet, the three points of intersection will be in the same straight line. Required the demonstration?

Solutions to these questions must come to hand (post-paid) by the First of November, 1806.

NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

III. MATHEMATICAL QUESTIONS.

To be answered in Number VIII.

I. QUESTION 181, by YANTO.

Bacchus caught Silenus asleep by the side of a full cask, he seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that, Silenus awakes, and drinks what Bacchus had left. Had they drank both together, it would have been emptied two hours sooner, and Bacchus would have drank only half of what he left for Silenus. Required the time in which they would empty the cask separately?

II. QUESTION 182, by AMICUS.

Two given beams, or rods, loaded with weights, being placed on a horizontal plane, so that their lower extremities may be opposed to each other, and the other extremities be supported by two parallel vertical planes; it is required to find their position when they are in equilibrio, and the pressure exerted against each plane?

III. QUESTION 183, by QUIDAM.

Given all the four sides, and the difference of the parts into which one of the angles is divided by a diagonal; to construct the trapezium and point out the limits.

IV. QUESTION 184, by Mr. JOHNSON, Birmingham.

A given quantity of gold is to be made into a cup in the form of a segment of a sphere, and n tenths of an inch in thickness: what are the dimensions of the cup when it is made so as to contain the greatest quantity possible?

V. QUESTION 185, by QUIDAM.

Let a string of a given length be fastened at one end to a point A in the circumference of a circle, and wound tightly about the circumference, and brought with the other end T to meet the diameter AB in T: what must be the diameter of the circle that the area BTP included by the right lines TP, TB, and the arc BP may be a maximum, TP being the tangent from T?

VI. QUESTION 186, by Mr. JOHN WHITLEY.

If an ellipse be circumscribed by a triangle, and lines be drawn from the angles of the triangle to the points of contact, meeting the ellipse again in three points; tangents drawn to the curve at these points will intersect each other in the said lines produced. Also, if the said tangents and the sides of the triangle be produced till they meet, the three points of intersection will be in the same straight line. Required the demonstration?

VII. QUESTION 187, by LAPUTIENSIS.

Let two spherical balls given in magnitude and weight, and having their surfaces perfectly polished, be put into a spherical vessel of a given diameter, whose internal surface is also perfectly polished: required the distances of the points of contact from the lowest point of the vessel, when the balls rest in equilibrio?

VIII. QUESTION 188, by Mr. IVORY.

Two points being given in the same vertical plane, but not in one horizontal line; it is required to determine the position of two inclined planes such, that the time of descent from the higher point to the lower, on these two planes, shall be less than on any other two inclined planes whatsoever?

IX. QUESTION 189, by Mr. CUNLIFFE.

Required the sum of the infinite series

$$\frac{1}{2} - \frac{1}{2} \times (\frac{1}{2}) + \frac{1}{4} \times (\frac{1}{2} + \frac{1}{2}) - \frac{1}{8} \times (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) + \frac{1}{16} \times (\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) - \&c?$$

The next four Questions are selected from the Cambridge University Calendar for 1806.

X. QUESTION 190, from the CAMB. UN. CAL.

A given cylindric vessel is supplied with water by a cock at a given rate. Then suppose, when the vessel is full, that an aperture of a given area is made in its bottom; what is the lowest point to which the surface of the water in the vessel will descend, and also the time of its descent; the influx of the water by the cock being supposed slower than the efflux when the vessel is full?

XI. QUESTION 191, from the CAMB. UN. CAL.

Bradley observed that every star passed the meridian farthest south when it came about six in the morning, whatever were its position with respect to the cardinal points of the ecliptic. Suppose then the place of a star to be given; on what day will it pass the meridian of London farthest to the south; and at what hour will it pass it on that day?

XII. QUESTION 192, from the CAMB. UN. CAL.

A perfectly flexible chain is wound round a cylinder supported with its axis parallel to the horizon. Then, if the weight and dimensions of the cylinder be given, and also the length and weight of the chain; it is required to determine the time in which the chain, impelled by the force of gravity, will unwind itself; a given length being unwound at the commencement of the motion?

XIII. QUESTION 193, from the CAMB. UN. CAL.

A cylindrical vessel, whose height is equal to its diameter, is filled with water; with what velocity must it be whirled round its axis, that half the water may be thrown out?

XIV. QUESTION 194, by AMICUS.

Of all the ellipses that can be inscribed in the same given quadrilateral, required that which approaches the nearest to a circle?

XV. QUESTION 195, *by Mr. I. R.*

A parabola, and a point within it, being given in position, it is required to describe another parabola, that shall pass through the given point, touch the given parabola in a given point, and have its axis parallel to the axis of the given parabola.

XVI. QUESTION 196, *by Mr. WILLIAM SLENDER.*

Let there be a right line and a point given by position in the same vertical plane: and suppose a flexible string of a given length, considered as without weight, to be fastened at one end to the given point, and a small heavy body or bead to slide freely thereon; required the locus of the bead, whilst the other end of the string is carried slowly along the right line given by position?

XVII. QUESTION 197, *by Mr. IVORY.*

Two points being given in a vertical plane, but not in the same horizontal line; it is required to determine the position of three inclined planes such, that (supposing the body to pass from one plane to another without any loss of velocity) the time of descent may be less than the time on any other three planes, when the line of descent is of a given length?

XVIII. QUESTION 198, *by MECHANICUS.*

Required the nature of the curve along which a heavy body descending by the force of gravity, with a given initial celerity, will press upon the curve at any point with a force reciprocally proportional to the radius of curvature at that point?

XIX. QUESTION 199, *by Mr. CUNLIFFE.*

Find a series of numbers, every one of which is divisible into three such parts, that if to the square of each part the product of the other two be added, the three sums thence arising shall all be rational squares?

XX. QUESTION 200, *by HYPATIA.*

Required the nature of the curve that is the locus of the point in which a perpendicular from the centre of an ellipse meets a tangent to the ellipse. Required also the rectification of the curve and its quadrature?

XXI. QUESTION 201, *by CURIOSUS.*

Find the shortest distance between two given points on the surface of a spheroid?

XXII. QUESTION 202, *by Mr. WALLACE, R. M. College.*

Find the length of a curve, the nature of which is expressed by the equation

$$e^y = \frac{e^x + 1}{e^x - 1},$$

where x and y denote the co-ordinates, and e the number of which the hyperbolic logarithm is unity?

XXIII. QUESTION 203, *by Mr. CUNLIFFE.*

To find three numbers such, that the sum of every two shall be a biquadratic, and the sum of all the three a square number?

XXIV. QUESTION 204, *by Mr. IVORY, R. M. College.*

If there be two concentric ellipses, similar to one another, and similarly posited, the proportion of the axes being that of $\sqrt{2}$ to 1; then if, from any point in the periphery of the inner ellipse, any right line be drawn to terminate in the periphery of the outer ellipse, the rectangle under the segments of that line, is equal to the square of that semi-diameter of the inner ellipse, which is parallel to the right line. Required the demonstration?

XXV. QUESTION 205, *by Mr. LOWRY.*

Of all semi-parabolas having the same area, required that along which a body will descend, by the force of gravity, in the least time possible?

XXVI. QUESTION 206, *by URSA MINOR.*

Suppose a cone to be cut by a plane, and the curved surface of the part cut off to be extended into a plane surface. Required the nature of the plane curve, formed by the common section of the cone and the cutting plane?

XXVII. QUESTION 207, *by Mr. P. R.*

Two magnetical poles being given in position; the force of each of which is supposed to be as the *mth.* power of the distance from it reciprocally; it is required to find a curve, in any point of which a needle (indefinitely short) being placed, its direction, when at rest, may be a tangent to the curve?

XXVIII. QUESTION 208, *by Mr. WALLACE, R. M. College.*

Shew that the rectification of the curve called the *line of sines* depends upon that of an ellipse, without employing the fluxionary calculus, or in any other way comparing the indefinitely little increments of the curves?

XXIX. QUESTION 209, *by Mr. IVORY.*

If straight lines be inflected from the extremities of the transverse axis of an ellipse, or hyperbola, to any point in the curve, and perpendiculars, drawn to the inflected lines from the same point, be produced to cut the same axis, the part of the axis intercepted by the perpendiculars, will be constantly of the same magnitude. Required the demonstration?

XXX. PRIZE QUESTION 210, *by Mr. LOWRY.*

In a given ellipse to inscribe a polygon of a given number of sides, so that each side may pass through a given point?

Solutions to these Questions must come to hand (post-paid) by the first of May 1807.

NEW SERIES OF THE MATHEMATICAL REPOSITORY.

IV. MATHEMATICAL QUESTIONS.

To be Answered in Number IX.

I. QUESTION 211, from the BIJA GONETA, or HINDOO ALGEBRA.

Find two integer square numbers whose sum and difference diminished by 1 shall both be squares?

"The *Bija Goneta* was written in Sanscrit by BHASKER ACHARI, a famous Hindoo Mathematician and Astronomer, who lived about the beginning of the 13th century of the Christian era: and was translated into Persian in 1634."

II. QUESTION 212, by INVESTIGATOR.

What sum should A, aged 30, pay on the event of his surviving B, aged 65, for an annuity of 1000*l.* during their joint lives, using the Northampton table of the probable duration of life, and allowing 5 per cent. interest of money?

III. QUESTION 213, by Mr. BAZLEY.

Find a circle whereof the diameter, and the chord, sine, and versed-sine of an arc of it, are all integers?

IV. QUESTION 214, by AMICUS.

The ends A, C of two given beams are moveable about two fixed points, and the other ends B, D are connected by a string of a given length which passes over a fixed pulley. It is required to determine the position of the beams when they are in equilibrium; the pulley and beams being in the same vertical plane?

V. QUESTION 215, by HYPATIA.

From A one end of AB, the diameter of a circle, draw any chord AC, and draw CD perpendicular to AB. Bisect the arch AC in E, and draw BE, meeting CD in G, and CA in F. The lines BG, GF shall be equal. Required the demonstration?

VI. QUESTION 216, by Mr. WALLACE, R. M. College.

Let a circle be given by position and let A and B be two given points; a point C may be found, such, that if any straight line be drawn through it, meeting the circle in D and E and AD, BD, also AE, BE be joined, there is a certain given space which is a mean proportional between $AD^2 + BD^2$ and $AE^2 + BE^2$. Required the demonstration?

VII. QUESTION 217, by Mr. LOWRY.

An ellipse, and a straight line, being given by position; a point may be found such, that if any straight line be drawn through that point to meet the ellipse in two points, and perpendiculars be drawn from those points to the straight line given by position, the rectangle of those perpendiculars shall always be equal to a certain given space?

VIII. QUESTION 218, by Mr. LOWRY.

If the vertical angle and sum of the sides of a plane triangle remain constant, what is the equation of the curve to which the base is always a tangent?

IX. QUESTION 219, by Mr. WALLACE.

Let AB, AC be two straight lines given by position, and P a given point: it is required to draw a line through P to meet the given lines in D and E, so that $AD \times AE$ may be to $DP \times PE$ in a given ratio.

X. QUESTION 230, by Mr. LOWRY.

If through a given point P on the surface of a sphere two great circles be drawn to intersect a small circle given by position in the points A, B and C, D; and the points AC, BD be joined by great circles which intersect in Q. What is the locus of the point Q?

XI. QUESTION 221, by Mr. LOWRY.

Find the perspective representation of a given spheroid placed on a horizontal plane; the relative positions of the eye, the vertical plane of the picture, and the spheroid being given?

XII. QUESTION 222, by Mr. BAZLEY.

It is required to determine the locus of the vertex of a plane triangle, whereof the radius of the inscribed circle, and difference of the angles at the base are given; the middle of the base being a fixed point in a straight line given by position?

XIII. QUESTION 223, by ———.

Let a circle be given by position, and let A and B be two given points, a point C may be found, such, that if any straight line be drawn through C to meet the circle in D and E, and AD, BD, AE, BE be joined, $AD^2 + DB^2 : AE^2 + EB^2 :: DC : CE$. Required the demonstration?

XIV. QUESTION 224, by Mr. IVORY.

Supposing x to denote any fraction less than unit, it is required to assign the value of the infinite product $(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \&c.$?

XV. QUESTION 225, by Mr. CUNLIFFE.

To find integer values for the sides of a trapezium that may be inscribed in a circle, such, that its diagonals, diameter of the circumscribing circle, and area, may all be expressed by whole numbers?

XVI. QUESTION 226, by Mr. CUNLIFFE.

Suppose a heavy hollow cylinder, of a uniform thickness, to roll freely along an inclined plane of a given length and inclination: having given the external diameter of the cylinder together with the time of rolling along the inclined plane, it is required from thence to determine the internal diameter of the cylinder?

XVII. QUESTION 227, by Mr. CUNLIFFE.

Required the sum of the infinite series

$$\frac{1}{1 \cdot 3^2} + \frac{2}{3 \cdot 5^2} + \frac{3}{5 \cdot 7^2} + \frac{4}{7 \cdot 9^2} + \frac{5}{9 \cdot 11^2} + \&c.$$

XVIII. QUESTION 228, by Mr. WALLACE.

Find the nature of a curve, such, that any normal shall have a given ratio to the segment of the axis intercepted between it and a given point?

XIX. QUESTION 229, by Mr. LOWRY.

Two equal bodies, connected by a flexible string, are placed on a smooth horizontal table, the string being stretched out in a direction parallel to the edge of the table. To the middle of this string, there is fastened another string, at the extremity of which is a given weight that hangs over the edge of the table, in a vertical direction, and, by its descent, communicates motion to the other bodies. Now, supposing the weights of the bodies and

and the length of the connecting string to be given; it is required to define the motion of the descending weight, and to ascertain its place at the end of any assigned time?

XX. QUESTION 220, by Mr. LOWRY.

A circle and a sphere being given by position, it is required to determine the perspective representation of the circle on the surface of the sphere, and to find the rectification of the curve; supposing the eye to be at the centre of the sphere?

XXI. QUESTION 231, by Mr. LOWRY.

If from two given points on the surface of a sphere, two great circles be drawn to any point D in the circumference of a small circle given by position, meeting it again in E and F: then the great circle joining the points E, F will either pass through a given point P; or a great circle drawn through E and P will cut off from the small circle EDF an arch FG of a given length?

XXII. QUESTION 232, by Mr. WALLACE.

Let F be the focus of a conic section, and P a given point in the axis. Draw FA to any point of the curve; take PB in the axis equal to FA and join AB. There is a straight line given by position which divides AB into parts having to each other a given ratio. Required the demonstration?

XXIII. QUESTION 233.

In the *Traité de Calcul Différentiel et de Calcul Intégral* of Bossut, the following theorem is given (but without demonstration) from the works of John Bernoulli.

"Let A₄B (fig. 95, pl. 4.) be the semi-circumference of a circle, of which AB, the diameter, is equal to the two semi-axes AC, BC of a given ellipse. Divide the semi-circumference A₄B into a number, 2, 4, 8, 16, &c. of equal parts, draw straight lines from the point C to all the points of division: take the arithmetical mean of all the straight lines drawn to the points of division denoted by odd numbers, and also the arithmetical mean of the sum of all the lines drawn to the points of the even numbers, and the half of AB; these two arithmetic means shall be the radii of two circles, the circumference of one of which shall be less than that of the ellipse, and the circumference of the other greater." Required the demonstration?

XXIV. QUESTION 234, by Mr. IVORY.

A point may be found within the base of every oblique cone, such, that if any plane be drawn through the vertex and the point found to cut the cone, then the vertical angle of the triangle, which is the common section of the plane and the cone, will be bisected by the right line drawn from the point found to the vertex of the cone?

XXV. QUESTION 235, by Mr. IVORY.

It is required to shew that the surface of a right cone on an elliptical base may be found by the rectification of the ellipse.

By a right cone on an elliptical base is meant such a cone as has an ellipse for its base, and likewise the right line, drawn from the vertex to the centre of the ellipse, perpendicular to the plane of the base?

XXVI. QUESTION 236, by Mr. IVORY.

From a point assumed without a cone, let two tangents be drawn to the same conic surface, or to the opposite surfaces:

and let a plane be drawn through the vertex of the cone and the two points of contact of the tangents: then, if a right line be drawn from the assumed point to cut the plane, and either of the conic surfaces in two points, that right line will be harmonically divided?

N. B. The demonstration of some properties of all the conic sections readily follows from this proposition

XXVII. QUESTION 237, by Mr. IVORY.

It is required to determine the position of the axes of the representation of a circle given by position in a horizontal plane: supposing the place of the eye to be given, and the plane of the picture perpendicular to the plane of the horizon?

XXVIII. QUESTION 238, by Mr. IVORY.

Let n be any whole positive number, and put

$$Q = \frac{2 \cdot 4 \cdot 6 \cdot 8 \dots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \dots (2n+1)}; \text{ then the whole value of the}$$

triple fluent $\int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{x^{2n+1} dx}{\sqrt{(1-x^2)}}$, which is generated while x increases from 0 to 1, is equal to

$$Q \times \left\{ \frac{1}{(2n+3)^2} + \frac{1}{(2n+5)^2} + \frac{1}{(2n+7)^2} + \frac{1}{(2n+9)^2} + \&c. \right\}?$$

N. B. Observe that the case $n=0$ is included by taking $Q=1$.

XXIX. QUESTION 239, by Mr. IVORY.

Let n be a whole positive number and put

$$P = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots 2n}; \text{ then the whole value of the}$$

triple fluent $\int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{x^{2n} dx}{\sqrt{(1-x^2)}}$, which is generated while x increases from 0 to 1, is equal to

$$P \times \left\{ \frac{1}{(2n+2)^2} + \frac{1}{(2n+4)^2} + \frac{1}{(2n+6)^2} + \frac{1}{(2n+8)^2} + \&c. \right\} \times \frac{\pi}{2}$$

where π is the semi-circumference to the radius 1?

N. B. Observe that the case $n=0$ is included by taking

$$P=1.$$

XXX. PRIZE QUESTION, 240, by Mr. IVORY.

Let a plane be drawn through the axis of an oblique cone, so as to be perpendicular to the plane of the base; and let a right line be drawn to bisect the vertical angle of the triangle which is the common section of the plane and the cone: then, if a second plane be drawn through this line (*viz.* the line which bisects the vertical angle of the triangle) to cut the cone, the difference of two portions of the conic surface between the two planes, and comprehended in opposite angles, may be determined by means of a circular area and a rectilineal space?

Solutions to these questions must come to hand (post-paid) by the First of May 1808.

ERRATA.

In page 4, vol. 2, part 2, line 11 from the bottom

$$\text{for } A \times \frac{x^2 u^2}{y^2} + D \times \frac{x^2 u^2}{z^2} \text{ read } A \times \frac{x^2 u^2}{y^2} + B u^2 + D \times \frac{x^2 u^2}{z^2}$$

NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

VOL. II. PART I.

MATHEMATICAL QUESTIONS.

ARTICLE I.

Solutions to Questions proposed in Number IV.

I. QUESTION 121, by SCOTICUS.

If α and β denote given arches, and

$$\sec. \alpha \sec. \varphi + \tan. \alpha \tan. \varphi = \sec. \beta.$$

Required the values of $\sec. \varphi$ and $\tan. \varphi$?

FIRST SOLUTION, by SCOTICUS, the Proposer,

Because

$$\sec. \alpha + \tan. \alpha = \frac{1}{\sec. \alpha - \tan. \alpha}$$

$$\sec. \varphi + \tan. \varphi = \frac{1}{\sec. \varphi - \tan. \varphi}.$$

By taking the product of these two equations and substituting $\sec. \beta$ for $\sec. \alpha \sec. \varphi + \tan. \alpha \tan. \varphi$ we have

$$\begin{aligned} & \sec. \beta + \sec. \alpha \tan. \varphi + \tan. \alpha \sec. \varphi \\ &= \frac{1}{\sec. \beta - (\sec. \alpha \tan. \varphi + \tan. \alpha \sec. \varphi)}. \end{aligned}$$

$$\text{But } \sec. \beta + \tan. \beta = \frac{1}{\sec. \beta - \tan. \beta}$$

therefore, by comparing this equation with the preceding one, it is manifest that

$$\sec. \alpha \tan. \varphi + \tan. \alpha \sec. \varphi = \tan. \beta.$$

VOL. II. PART I.

A

By

By taking the sum and difference of this equation, and the equation

$$\sec. \alpha \sec. \phi + \tan. \alpha \tan. \phi = \sec. \beta,$$

and resolving the result into factors, we find

$$(\sec. \alpha + \tan. \alpha) (\sec. \phi + \tan. \phi) = \sec. \beta + \tan. \beta$$

$$(\sec. \alpha - \tan. \alpha) (\sec. \phi - \tan. \phi) = \sec. \beta - \tan. \beta.$$

$$\text{Hence, } \sec. \phi + \tan. \phi = \frac{\sec. \beta + \tan. \beta}{\sec. \alpha + \tan. \alpha};$$

$$\sec. \phi - \tan. \phi = \frac{\sec. \beta - \tan. \beta}{\sec. \alpha - \tan. \alpha};$$

$$\text{and } \sec. \phi = \sec. \alpha \sec. \beta - \tan. \alpha \tan. \beta;$$

$$\tan. \phi = \sec. \alpha \tan. \beta - \tan. \alpha \sec. \beta.$$

SECOND SOLUTION, by Mr. JOHN DAWES.

Put $a = \tan. \alpha$, $b = \sec. \beta$, and $x = \tan. \phi$, radius 1: then $\sqrt{1+a^2} = \sec. \alpha$, $\sqrt{b^2-1} = \tan. \beta$, and $\sqrt{1+x^2} = \sec. \phi$. These values being written in the proposed expression, it becomes

$$\sqrt{1+a^2} \sqrt{1+x^2} + ax = b,$$

and by transposition

$$\sqrt{1+a^2} \sqrt{1+x^2} = b - ax; \dots\dots\dots(q)$$

squaring both sides

$$(1+a^2)(1+x^2) = (b-ax)^2;$$

$$\text{that is, } 1+a^2+x^2+a^2x^2 = b^2 - 2abx + a^2x^2$$

$$\text{whence } x^2 + 2abx = b^2 - (1+a^2),$$

$$\text{and } x = \sqrt{(1+a^2)(b^2-1)} - ab;$$

$$\text{that is, } \tan. \phi = \sec. \alpha \tan. \beta - \sec. \beta \tan. \alpha.$$

Again from the equation q

$$\sqrt{1+a^2} \sqrt{1+x^2} = b - ax = b + ba^2 - a \sqrt{(1+a^2)(b^2-1)}$$

and dividing by $\sqrt{1+a^2}$,

$$\sqrt{1+x^2} = b \sqrt{1+a^2} - a \sqrt{b^2-1};$$

$$\text{that is, } \sec. \phi = \sec. \beta \sec. \alpha - \tan. \beta \tan. \alpha.$$

Q. E. I.

COR. The subsequent equations are easily obtained from the preceding, *viz.*

$$\sec. \alpha = \sec. \beta \sec. \phi \pm \tan. \beta \tan. \phi.$$

$$\tan. \alpha = \sec. \beta \tan. \phi \pm \sec. \phi \tan. \beta.$$

$$\tan. \beta = \sec. \alpha \tan. \phi \pm \sec. \phi \tan. \alpha.$$

Ingenious Solutions to this Question were also received from Messrs. Amicus, Rev. J. Toplis, Johnson, and J. W. of Leeds.

II. QUESTION 122, *by* AMICUS.

If two tangents PE, PF be drawn to a conic section, and the line EF joining the points of contact be produced to meet the directrix in G; then if PI be drawn through the focus to meet EF in I, EI will be to IF as EG to FG, required the demonstration?

SOLUTION, *by* Mr. JOHN JOHNSON, *Birmingham.*

Fig. 1, pl. 1. Let f be the focus; join Ef , Ff , and draw EH , FK perpendicular to the directrix.

It is a property of the conic sections, now pretty generally known, that the angle Efp is equal to the angle Ffp , and therefore the angle Efl is equal to the angle Ffl , fl bisects the angle EfF ; wherefore $Ef : fF :: EI : FI$. Again by the property of the directrix $Ef : fF :: EH : FK :: EG : FG$, and comparing this proportion with the former we have $EI : IF :: EG : FG$.
Q. E. D.

III. QUESTION 123, *by* Mr. JAMES WHALLEY, *Bolton.*

To construct a trapezium there is given one of the sides, both the diagonals and the angle formed by their intersection, and the angle made by either diagonal and side opposite to the given one?

FIRST SOLUTION, *by* Mr. W. WALLACE, *R. M. College.*

Suppose ABCD (fig. 2, pl. 1.) to be the quadrilateral required, then by hypothesis AB one of its sides, and AC, BD, its two diagonals are given, also the triangle DEC, formed by the diagonals, and DC the opposite side, is given in species.

Produce AB, CD to meet in G, and draw AH parallel to the diagonal BD, then $BG : GA :: BD : AH$, but the ratio of BD to AH is compounded of the ratios of BD to AC and of AC to AH, that is of BD to AC and of EC to ED, both of which ratios are given since BD and AC are given lines and the triangle EDC is given in species, therefore the ratio of BG to GA is given, and AB being given, AG and BG are both given. Now in the triangle ACG the base AG, the vertical angle ACG and AC one of the sides are known, therefore the triangle may be easily constructed, and hence the point C will be known, any this point being determined all the rest may be found of course.

SECOND SOLUTION, *by* AMICUS.

Fig. 2, pl. 1. Take AC equal to one of the given diagonals, and draw AH, CH to make the angles CAH, ACH, equal to the angles made by the intersection of the diagonals and the other given angle respectively; produce AH, if necessary, till AF be equal to the other given diagonal, and from F to the line CH apply FD equal to the given side; complete the parallelogram FDBA and join BC; then ABCD is the trapezium required.

For since ABDF is a parallelogram, AB is equal to FD the given side, and BD is equal to AF one of the given diagonals, also AC is equal to the other diagonal; and the angles ACD, DEC (or HAC, because DE and HA are parallel) are equal to the given ones.

THIRD SOLUTION, *by* Mr. CUNLIFFE, *R. M. College.*

Fig. 2, pl. 1. Let CDBA represent the required trapezium, AB being the given side; draw BG' parallel to AC and produce DC to meet it in G'; also draw AS parallel to DG'. Then in the triangle DBG' the angles BDG' and DBG' are both given together with the side or diagonal DB, whence the triangle itself becomes entirely known.

The figure CASG' is a parallelogram by the construction, therefore GS = AC a given line, consequently S is a given point, and SA a right line given by position: whence the following construction is pretty obvious.

Having drawn SA, from B apply BA thereto equal to the given side, draw AC parallel to BG' meeting DG' in C, join CB, DA, and DCBA is the required trapezium, as is evident from the analysis.

This Question was likewise answered by Messrs. Bazley, Johnson, and Toplis.

IV. QUESTION 124, *by* SENEX:

Let there be two given points in the circumference of a circle given in magnitude and position: a circle may be found, given also in magnitude and position, such, that if from any point in its circumference straight lines be drawn to the given points, and a tangent to the given circle; the square of the tangent shall have to the rectangle of the lines drawn to the given points a given ratio.

SOLUTION,

SOLUTION, by Mr. BAZLEY.

Let A and B (fig. 3, pl. 1.) be the given points, ABED the given circle, C one of the points in the periphery of the circle which is to be found; draw the tangent CD and join AC, BC, and AB: let E be the point where AC cuts the given circle, and join BE.

Then by hypothesis, the ratio of $AC \times CB : CD^2 = AC \times CE$ is given; that is, the ratio of CB to CE is given; but since the circle ABED and the chord AB are given, therefore the angle AEB is given, and its supplement BEC; consequently the triangle BEC is given in species, and therefore the angle ACB standing on the given line AB is given. Wherefore, the point C is placed in the circumference of a circle described on the given line AB, to contain the given angle ACB.

Messrs. Amicus, Cunliffe, Johnson and I. W. of Leeds favoured us with Solutions to this Question.

V. QUESTION 125, by Mr. JOSEPH BREWER, Preston.

Two right lines being given by position, a circle may be found, such, that if another circle be described upon any radius thereof as a diameter, the chord of the arc of this latter circle intercepted by the lines given by position shall be of a given length.

SOLUTION, by Mr. BAZLEY.

Suppose AB, AC (fig. 4, pl. 1.) to be the two straight lines given by position, apply DE of the given length perpendicular to AB and meeting AC in E; with the centre A and distance AE describe the circle ER, which is that which was to be found. For, on AE as a diameter, describe a circle, which will pass through D, because the angle D is a right one. Draw any other radius AR of the circle RE, whereon, as a diameter, describe the circle APQR, meeting AB in Q and AC in P, and join PQ. Then in the equal circles ADE, APQR, the chords ED, PQ, subtending the same angle BAC, are equal; that is, $PQ = DE =$ the given length.

Messrs. Johnson and J. W. of Leeds also sent answers to this Question.

VI. QUESTION 126, by Mr. J. JOHNSON.

Given the position of the centres of gravity of the three triangles, formed by lines drawn from the angular points of the triangle to the centre of the inscribed circle, to determine the position of the triangle?

FIRST

FIRST SOLUTION, *by Mr. BAZLEY.*

Let ABC (fig. 5, pl. 1.) be the required triangle; P, Q, R, the given points, and O the centre of the inscribed circle. Draw OPS meeting AB in S; then it is well known that S is the middle of AB and $SP = \frac{1}{3}SO$; through P draw DPE parallel to AB meeting OA, OB in D and E: then $AD = \frac{1}{3}AO$, and if DF be drawn parallel to AC meeting OC in F, CF is $= \frac{1}{3}CO$, and DF passes through Q. Again, join FE, which passes through R, because $CF = \frac{1}{3}CO$ and $BE = \frac{1}{3}BO$; moreover P, Q, R, are evidently the middle points of DE, DF, FE. Hence DPRQ, EPQR and FQPR are parallelograms; and therefore we have this

CONSTRUCTION.

Having joined P, Q, R, draw DPE parallel to QR taking PD, PE each equal to QR, and draw DQF, ERF; also draw DO, EO, bisecting the angles D and E and meeting in O; produce OD to A so that $OA : OD :: 3 : 2$, and in the same ratio produce OE to B; lastly draw AC, BC (meeting in C) parallel to DF, EF, and the thing is done: the demonstration is plain from the analysis.

SECOND SOLUTION, *by the Rev. J. TOPLIS.*

Let ABC (fig. 6, pl. 1.) be the triangle whose position is required, AOB, AOC, and BOC the three triangles formed by lines drawn from the angular points to the centre of the inscribed circle, and G, H, F their centres of gravity. Join GH, HF, FG, and let the lines AGE, BFD, CHP be drawn through the points G, F, H, meeting the lines OB, OC, OA, and (by the property of the centre of gravity) bisecting them in E, D, and P; then it is evident that the lines joining the points A, D; B, P; C, E, will pass through the points H, G, F respectively; and by a well known property of the centre of gravity, BP is equal to 3GP, PC to 3PH; therefore GH and BC are parallel, and BC equal to 3GH. In a similar manner it may be proved that AC is parallel to GF and AB to FH and that AC is equal to 3GF and AB to 3FH; wherefore the triangle ACB is similar to, and equal to thrice the given triangle FGH. Let Q be the centre of the circle inscribed in the triangle FGH; bisect the lines GQ, HQ in K and I, and draw GI, HK, FK, FI; then it is obvious that the triangle GHQ is similar to the triangle BCO; and since GQ, CO, are bisected in K and D, and BC equal

equal to $3GH$, it is evident that BD ($3FD$) is equal to $3HK$, that is, FD is equal to HK ; and in a similar way it is shewn that FE is equal to GI , DH to FK , and EG to FI , wherefore HK , GI , FK , FI are given lines; hence if FD , DH , FE , GE be taken equal to the given lines, the points D , E , will be determined, and if DH , DF , EF be produced till DA is equal to $3DH$, BD to $3DF$, and EC to $3EF$, the position of the angular points of the triangle become known.

An ingenious Solution was received from Mr. Johnson, the Proposer.

VII. QUESTION 127, by HYPATIA.

Suppose $u = \frac{a^{a-n} - x^{x-n}}{a-x}$, what is the value of u when $x = a$?

SOLUTION, by the Rev. J. TOPLIS.

The fluxion of $a^{a-n} - x^{x-n}$ is $-\log. x \times x^{x-n} \dot{x} - \frac{x^{x-n}(x-n)\dot{x}}{x}$, and the fluxion of $a-x$ is $-\dot{x}$; therefore the fluxion of the numerator divided by the fluxion of the denominator is $x^{x-n} \times \log. x + x^{x-n} - n x^{x-n-1} =$ (when x becomes a) $a^{a-n} \times (\log. a + 1 - \frac{n}{a})$.

The values of fractions of the form $\frac{1}{x}$ are very amply treated upon in the first vol. of the *Traité du Calcul Différentiel et Integral* par S. F. Lacroix, page 241.

VIII. QUESTION 128, by URSA MINOR.

Let AB , AC , be two straight lines given by position, and let a straight line DE meet them in D and E , so that $AD + AE$ is equal to a given line, and let DE be divided at V , so that DV may have to VE a given ratio. Required the locus of the point V ?

FIRST SOLUTION, by AMICUS.

Analysis. In fig. 7, pl. 1, take $AH =$ the given sum and make HI to IA in the given ratio of DV to VE ; draw IV to meet

meet AC in F, and let PV, QV be drawn parallel to the lines AC, AB, respectively; then because AD + AE is = AH, DH is = AE, and by similar triangles AP : AD :: (VE : DE) AI : AH, and by permutation and division IP : AI :: HD = AE : AH; and again by similar triangles and permutation AQ = PV : IH :: AE : AH; therefore IP : PV :: (IA : AF) IA : IH which is a given ratio, and shews that AF is = to IH, therefore the locus of the point V is the straight line IF given by position. Hence this construction. Divide the given sum AH into two parts AI, HI, having the given ratio, make AF = IH and the line joining the points I, F, is the locus required.

It is evident from hence that if two bodies whose weights are proportional to the segments VE, VD, be placed on the inclined planes AB, AC, and connected by a string whose length is AH and which passes over the common section of the planes at A, the locus of the centre of gravity of the two bodies will be the straight line IF.

If the difference, instead of the sum of AD, AE, had been given the locus would also be a straight line given by position; for if hA be taken equal to the given difference, the analysis and construction will be precisely the same as above.

But the problem may be made more general, for if R and S be any given lines and the sum or difference of the rectangles AD x R, AE x S be equal to a given space, then the locus of the point V will still be a straight line.

For let AK x R be equal to the given space, and divide AK at I so that AI may be to IK in the given ratio of VE to DV; then because AD x R + AE x S = AK x R, we have AE x S = DK x R. And by similar triangles &c. as in the preceding Analysis,

IP : AI :: KD : AK :: KD x R : AK x R;
also AQ = PV : KI :: AE : AK :: AE x S : AK x S;
therefore IP : PV :: AI : IF) :: AI x S : KI x R,
a given ratio, and it is obvious that AF is = to KI therefore the locus of the point V is a straight line IF as before.

The same Algebraically.

Draw VP parallel to AC and VQ to AB. Put s = the given sum of AD, AE; x = AP = VQ, y = PV = AQ, and m : n the given ratio of DV to VE, then by similar triangles

$$n : m :: x : \frac{mx}{n} = DP,$$

$$m : n :: y : \frac{ny}{n} = QE,$$

$$\text{hence } x + \frac{mx}{n} = \frac{m+n}{n} \cdot x = AD,$$

$$\text{and } y + \frac{ny}{m} = \frac{m+n}{m} \cdot y = AE;$$

therefore $\frac{m+n}{n} \cdot x + \frac{m+n}{m} \cdot y = s$, an equation to a straight line.

$$\text{Now when } y = 0, x \text{ is } = \frac{sn}{m+n},$$

and when $x = 0$, y is $= \frac{sm}{m+n}$. And these values of x and y determine the points where the locus meets the straight lines AB, AC: hence if AI be taken $= \frac{sn}{m+n}$ and AF $= \frac{sm}{m+n}$ the straight line IF will be the locus required.

SECOND SOLUTION, by Mr. BAZLEY.

(Fig. 8, pl. 1.) Divide the given sum of AD and AE in the given ratio of DV to VE; take on AC, AI, equal to the first part, and on AB take AF, equal to the other part, and join IF, which is the required locus. For, take on AC any distance IE, and on FA take FD equal IE (the one towards the point A, the other from it); let DE meet IF in V, and draw DG parallel to AC, to meet IF in G. Then it is plain that $AD + AE = AF + AI$ = the given sum by construction; and since the triangles AFI, DFG; DVG, EVI, are similar; therefore $EV : VD :: IE :: DF : DG :: AF : AI$, that is; $EV : VD :: AF : AI$, which is the given ratio, by construction.

Solutions were also received from Messrs. Cunliffe, Johnson, Toplis, and J. W. of Leeds.

IX. QUESTION 129, by Mr. CUNLIFFE, R. M. College.

To find two rational numbers such, that if a given number n be subtracted from either of them, or from their sum or difference, the four remainders shall all be rational squares.

SOLUTION, by Mr. CUNLIFFE, the Proposer.

Let $x^2 + n$ and $y^2 + n$ denote the two numbers, by which notation the two first conditions of the question will evidently be satisfied: it therefore only remains to determine such values of x and y as shall make the two remaining conditions squares, viz. $x^2 + y^2 + n$ and $x^2 - y^2 - n$, both squares.

Assume $x + v$ for the root of the first, that is, put $x^2 + y^2 + n = (x + v)^2 = x^2 + 2xv + v^2$, which gives $x = \frac{y^2 + n - v^2}{2v}$, by means of which $x^2 - y^2 - n = \frac{(y^2 + n - v^2)^2}{4v^2} - y^2 - n = \frac{y^4 - 6v^2y^2 + 2ny^2 - 6nv^2 + v^4 + n^2}{4v^2} =$ a square;

therefore $y^4 - 6v^2y^2 + 2ny^2 - 6nv^2 + v^4 + n^2 =$ a square; Assume $s - y^2$ for the root of the preceding square, that is, put $y^4 - 6v^2y^2 + 2ny^2 - 6nv^2 + v^4 + n^2 = (s - y^2)^2 =$

$s^2 - 2sy^2 + y^4$; from whence $y^2 = \frac{s^2 - v^4 + 6nv^2 - n^2}{2 \times (s + n - 3v^2)} =$ a square;

take $s = n - v^2$, then $y^2 = \frac{s^2 - v^4 + 6nv^2 - n^2}{2 \times (s + n - 3v^2)} = \frac{nv^2}{n - 2v^2} = \frac{n^2v^2}{n^2 - 2nv^2} =$ a square; therefore $n^2 - 2nv^2$ must be a square.

Put $n^2 - 2nv^2 = (rv - n)^2 = r^2v^2 - 2rvn + n^2$, whence $v = \frac{2rn}{r^2 + 2n}$; and $y = \frac{nv}{rv - n} = \frac{2rn}{r^2 - 2n}$; and from what

has been deduced $x = \frac{y^2 + n - v^2}{2v} = \frac{3r^4n^2 + (r^4 - 4n^2)^2}{4r \times (r^2 + 2n) \times r^2 - 2n^2}$ where r may be taken at pleasure.

Example 1. Take $r = 1$, and suppose the given number $n = 1$; then $y = \frac{2rn}{r^2 - 2n} = -2$, and $x = \frac{3r^4n^2 + (r^4 - 4n^2)^2}{4r \times (r^2 + 2n) \times r^2 - 2n^2} = \frac{41}{12}$; whence $x^2 + 1 = \frac{1825}{144}$ and $y^2 + 1 = 5$, which two numbers will answer.

Example

Example 2. Take $r = 1$, and $n = 2$, then $y = -\frac{4}{3}$, and $x = \frac{353}{180}$; whence $x^2 + 2 = \frac{189409}{32400}$; and $y^2 + 2 = \frac{84}{9} = \frac{34 \times 3600}{9 \times 3600} = \frac{122400}{32400}$.

X. QUESTION 130, *by* TRIGONOMETRICUS.

It is required to demonstrate that the tangent of an arch of 9° is equal to $1 + \sqrt{5} - \sqrt{(5 + 2\sqrt{5})}$, the radius being 1?

FIRST SOLUTION, *by* Mr. J. JOHNSON.

Put s for the sine and c for the cosine of 18° ; then (Emerson's Trigonometry, page 12,) the tangent of half the arc, or of 9° is $= \sqrt{\frac{1-c}{1+c}} = \sqrt{\frac{(1-c)^2}{1-c^2}} = \frac{1-c}{s}$. But it is well known that the sine of 18° is $= \frac{\sqrt{5}-1}{4}$, therefore its cosine is $= \sqrt{1-s^2} = \frac{1}{4}\sqrt{10+2\sqrt{5}}$, and $\frac{1-c}{s} = \left\{ 1 - \frac{1}{4}\sqrt{10+2\sqrt{5}} \right\} \div \frac{\sqrt{5}-1}{4} = \frac{4 - \sqrt{10+2\sqrt{5}}}{\sqrt{5}-1} = 1 + \sqrt{5} - \sqrt{(5+2\sqrt{5})} = \text{tangent of } 9^\circ \text{ as required.}$

SECOND SOLUTION, *by* Mr. CUNLIFFE.

Let t denote the tangent of an arch A , radius 1; then the writers on trigonometry have shewn that the tangent of $5A = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$; and this formula will also express the tangents of the following arches *viz.* $5A \pm 180$ and $5A \pm 2 \times 180$.

Suppose $5A = 45^\circ$, then the tangent of $5A = 1 = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$, whence $t^5 - 5t^4 - 10t^3 + 10t^2 + 5t - 1 = 0$, the five roots of which equation are the tangents of 9° , 45° , -27° , 81° and -63° . And, since the tangent of 45° is a root of the above equation, therefore $t - 1$ must be a divisor thereof, and the quotient

thence arising will be $t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$, completing the square by adding $20t^2$ to both sides $t^4 - 4t^3 + 6t^2 - 4t + 1 = 20t^2 = 4t^2 \times 5$; taking the roots $t^2 - 2t + 1 = \pm 2t\sqrt{5}$, or $t^2 - 2t \times (1 \pm \sqrt{5}) = -1$; again completing the square $t^2 - 2t \times (1 \pm \sqrt{5}) + (1 \pm \sqrt{5})^2 = 5 \pm 2\sqrt{5}$; taking the roots, &c. $t = 1 \pm \sqrt{5} \pm \sqrt{(5 \pm 2\sqrt{5})}$, and the two positive values of t are $1 + \sqrt{5} - \sqrt{(5 + 2\sqrt{5})}$ and $1 + \sqrt{5} + \sqrt{(5 + 2\sqrt{5})}$, the former of which must therefore be the tangent of 9° , and the latter the tangent of 81° .

Ingenious Solutions were also received from Messrs. Amicus, Dawes, and J. W. of Leeds.

The next Six Questions are selected from the Cambridge University Calendars for 1804 and 1805.

XI. QUESTION 131, from the CAMB. UN. CAL.

A given parabolic conoid is put into a fluid, and when it is in a position of permanent equilibrium the base is wholly extant, and inclined to the horizon; having given its specific gravity, relatively to that of the fluid, it is required to determine the dimensions of the part immersed?

FIRST SOLUTION, by Mr. LOWRY.

LEMMA. A PROBLEM. Fig. 9. Pl. 1.

To find the centre of gravity of an oblique segment of a parabolic conoid.

It is demonstrated at prop. 1, sect. 4, part 3, of Dr. Hutton's Mensuration, that the section made by a plane cutting a paraboloid oblique to the axis will be always an ellipse, and that all the parallel sections will be like and similar figures. Hence, if AB be the transverse axis of the elliptic base, and LR a diameter to AB, the sections parallel to the base will be similar to the section ACB and have their centres in the diameter LR; and therefore the centre of gravity of the solid will also be in that diameter. Put $a =$ the base ACB, $b =$ the diameter LR, $s =$ the sine of the angle LRB, $x =$ any variable abscissa LD, whose ordinate is ab; o the centre of gravity of the segment aLb and O that of ALB. Then $b : x :: a : \frac{ax}{b} =$ the area of the

section at D, and $\frac{asx\dot{x}}{b}$ will be the fluxion of the solid aLb; and therefore

therefore, by the common formula for finding the centre of gravity

$$Lo = \frac{\int \frac{as}{b} x^2 \dot{x}}{\int \frac{as}{b} x \dot{x}} = \frac{\int x^2 \dot{x}}{\int x \dot{x}} = \frac{2x^3}{3x^2} = \frac{2}{3} x.$$

When x is equal to b , then Lo is equal to $\frac{2}{3} b$.

Now let ABC (fig. 10. pl. 1.) be a section of the conoid made by a plane passing through the axis HC, and through the point A in the base which is nearest to the surface of the fluid. Let EF be the line where the plane of the fluid meets the section, and EFC the part of the body immersed in the fluid; then, if G be the centre of gravity of the whole conoid, and O that of the immersed part EFC, the body will be in a position of permanent equilibrium when the line joining the points G, O, is perpendicular to EF the surface of the fluid; and the content of the immersed part EFC, will be to that of the whole conoid, as the specific gravity of the body to that of the fluid; therefore if s be the content of the conoid, and 1 to m the relative proportion of the specific gravities of the solid and fluid, then $\frac{s}{m}$ will be the content of the immersed part EFC;

and, to determine its dimensions, draw Eb parallel to AB meeting the axis in K, and let PQ, the diameter to the ordinate EF, meet Eb in D, and draw FN and GL perpendicular to Eb and PD; then PQ being parallel to KC, FD will be perpendicular to Eb, and because EP = PF, ED will be = DN and Nb = 2DK.

Put p = the parameter of the axis, and n = the area of a circle whose diameter is unity; then the content of the immersed segment EFQ is equal to a cylinder on the circular base EN

and altitude half the diameter PQ, that is, = to $EN^2 \cdot \frac{PQ}{2} \cdot n$

(pa. 378, cor. 2, of the work referred to in the lemma.) And by Emerson's conic sections, prop. 19, $EN^2 = 4ED^2 = PQ \cdot 4p$,

or $PQ = \frac{EN^2}{4p}$, therefore the content of the segment is = $\frac{EN^4}{4p}$

$\cdot \frac{n}{2} = \frac{s}{m}$ and $EN = \sqrt[4]{\frac{8ps}{nm}}$, a given quantity, for which put

$2a$, and then ED will be = to a . And because $PQ \cdot 4p = EN^2$,

PQ will be = to $\frac{1}{4} p \sqrt{\left(\frac{8ps}{nm}\right)} = \sqrt{\left(\frac{s}{2pnm}\right)}$, which denote

by $3b$. Now, when G is the centre of gravity of the conoid,

GC

GC is $= \frac{2}{3}CH = c$; and when O is the centre of gravity of the segment, OQ is $= \frac{2}{3}PQ$ (by the lemma), and $OP = \frac{1}{3}PQ = b$.

Let GO meet EF in I, then by Emerson's Conics, prop. 4.
 $p : EN :: Nb = 2DK : NF = 2DP$, or
 $p : 2ED :: DK : DP$. But the right angled triangles EDP, OPI, OLG being similar, we have
 $ED : OL :: PD : LG = DK$, and ex æquo, $p : 2OL :: DK : DK$; therefore OL is given $= \frac{1}{2}p$. Draw QT parallel to LG meeting the axis in T; then because OQ is $= 2b$, $GT = LQ$ is given $= \frac{1}{2}p + 2b$, and therefore TC is $= GC - GT = c - \frac{1}{2}p - 2b$; and by the property of the curve $QT^2 = (DK^2) = p \times TC = p(c - \frac{1}{2}p - 2b)$, therefore $DK = \sqrt{pc - \frac{1}{2}p^2 - 2bp}$, a given quantity; from whence the ordinate EK and the abscissa KC become known, and the dimensions of the immersed segment are then easily determined.

The position of EF may also be found by a very simple construction: for having taken GT on the axis CG equal to the given length, and drawn QT perpendicular to GC meeting the curve in Q, let QS and QR be taken on QT produced, each equal to ED, or half EN; then the lines SE, RF being drawn parallel to the axis, will meet the curve in the points E and F required.

SECOND SOLUTION, by Mr. CUNLIFFE.

Let ADB (fig. 11, pl. 1.) represent a section of the given parabolic conoid through its axis CD; also let LDH represent a section of the immersed part, lying in the same plane with the section ADB. Draw Lm, Hn and pd parallel to the axis CD; the latter, viz. pd bisecting LH in s; also draw Hqr and Lt parallel to AB.

Let G, in CD, be the centre of gravity of the whole parabolic conoid ADB; also let g, in pd, be the centre of gravity of the immersed segment LDH.

Then it is well known that when the floating body is in a position of permanent equilibrium, the right line joining gG will be perpendicular to LH, the surface of the fluid. Draw Gv parallel to AB meeting pd in v, and in the last mentioned circumstance the triangles Gvg and sHq will be similar.

Put $AC = BC = b$, $CD = a$, $Cn = rH = y$, $Cm = Lt = u$; then $mn = y + u$, $qH = \frac{1}{2}(y + u)$ and $Cp = Gv = \frac{1}{2}(y - u)$; and by the known property of the parabola,

$b^2 :$

$b^2 : a :: y^2 : \frac{ay^3}{b^2} = rD$; and in like manner $\frac{au^3}{b^2} = tD$;

whence $rD - tD = rt = \frac{a}{b^2} \times (y^3 - u^3)$, and $qs = \frac{1}{2}rt = \frac{a}{2b^2} \times (y^3 - u^3)$; and by the similar triangles Gvg , sHq , we have $qs : qH :: Gv : vg$, whence $vg = \frac{qH \times Gv}{qs} = \frac{b^2}{2a}$.

Again, by the known property of the parabola, $CB^2 : CD :: qH^2 : sd$, whence $sd = \frac{CD \times qH^2}{CB^2} = \frac{a \times (y+u)^2}{4b^2}$, and it is well known that $sg = \frac{1}{3}sd = \frac{a \times (y+u)^2}{12b^2}$;

Hence, $Cr = CD - rD = a - \frac{ay^2}{b^2} = \frac{a \times (b^2 - y^2)}{b^2}$,

$ps = Cr + qs = \frac{a \times (b^2 - y^2)}{b^2} + \frac{a \times (y^3 - u^3)}{2b^2} = \frac{a}{2b^2} \times \{2b^3 - (y^3 + u^3)\}$,

and $pg = ps + sg = \frac{a}{2b^2} \times \{2b^3 - (y^3 + u^3)\} + \frac{a}{12b^2} \times (y+u)^2$
 $= \frac{a}{12b^2} \times \{12b^3 - 6(y^3 + u^3) + (y+u)^2\}$.

But $pg = ps + sg = pv + vg = CG + vg$, that is,

$\frac{a}{12b^2} \times \{12b^3 - 6(y^3 + u^3) + (y+u)^2\} = \frac{a}{3} + \frac{b^2}{2a} = \frac{2a^2 + 3b^2}{6a}$.

Whence $12b^3 - 6(y^3 + u^3) + (y+u)^2 = \frac{2b^2}{a^2} \times (2a^3 + 3b^2) (1)$.

By Hutton's Mensuration part 3rd. sect. 6th. $4 \times BC^2 \times DC \times \frac{q}{2}$ expresses the content of the parabolic conoid ADB; and

$\frac{mn^4 \times DC \times \frac{1}{2}q}{4 \times BC^2} =$ the content of the oblique segment LHD,

where $q = .7854$.

Then by the principles of hydrostatics,

$4 \times BC^2 \times DC \times \frac{q}{2} : \frac{mn^4 \times DC \times \frac{1}{2}q}{4 \times BC^2} :: m : n$; m and n be-

ing the specific gravities of the fluid and floating body; whence

mn^4

$$mn^4 = 16 \times BC^4 \times \frac{n}{m}, \text{ or } mn = 2BC \sqrt[4]{\frac{n}{m}}; \text{ that is, } y + u \\ = 2b \sqrt[4]{\frac{n}{m}}, \text{ or } (y + u)^2 = 4b^2 \sqrt{\frac{n}{m}} \dots\dots\dots (2).$$

Now from the equations 1 and 2 the value of y may be found by a quadratic equation.

COR. From what is deduced in the solution, it appears that the centre of gravity of the immersed segment LDH is always at a given perpendicular distance from the base of the parabolic conoid, whatever are the relative densities of the fluid and floating body.

$$\text{For } pg = \frac{a}{3} + \frac{b^2}{2a} = \frac{2a^2 + 3b^2}{6a} = \text{a given length.}$$

XII. QUESTION 132, from the CAMB. UN. CAL.

Find the centre of gravity of a cylindrical portion of the atmosphere measured from the earth's surface, the force of gravity being constant?

FIRST SOLUTION, by Mr. LOWRY.

Fig. 12, pl. 1. Let ABC be the base, and DC the axis of a cylindrical portion ABGH of the atmosphere. Imagine this cylinder to be divided into an indefinite number of thin parts, or laminæ of equal thickness; then it is evident that the quantity of matter in each of these laminæ will be proportional to the density of the atmosphere at the respective altitudes. Conceive a solid EAFB, symmetrical to the axis DC, to be described on the same base as the cylinder, and composed of particles of matter of the same density as the air at the earth's surface and divided into laminæ of the same thickness as those into which the cylinder is divided; then if the nature of the solid be such, that the quantity of matter in each of the laminæ of the solid is equal to the quantity of matter in each of the corresponding laminæ of the cylinder, it is evident that the solid EAFB will have the same centre of gravity S, as the cylinder AGHB. Put a = the area of the base, x = the altitude DC, d = the density of the air at the earth's surface, and y = the density at the height DC. Then the quantity of matter in the laminæ GHK is = ay , which, by hypothesis, is = to $d \times (\text{area EDF})$, therefore $EDF = \frac{a}{d} \times y$, and the flux-

ion

ion of the quantity of matter in the solid $= \frac{a}{d} \times y\dot{x}$; and by the common formula for finding the centre of gravity of a solid, we have

$$CS = \frac{\int \frac{a}{d} \times y\dot{x}}{\int \frac{a}{d} \times \dot{x}} = \frac{\int y\dot{x}}{\int \dot{x}}.$$

Now, the relation between the density and altitude being expressed by the equation $\frac{x}{n} = \log. \frac{d}{y}$, where n is $= 27819$ feet, the height of a homogenous atmosphere of the same density as the air at the earth's surface, we have $\dot{x} = -n \times \frac{\dot{y}}{y}$, and

$$CS = \frac{\int y\dot{x}}{\int \dot{x}} = \frac{\int -nxy}{\int -ny} = \frac{C - nxy + ny\dot{x}}{c - ny} = \frac{C - nxy - n^2y}{c - ny}.$$

But when y is $= d$, x is $= 0$, and then $C - n^2d = 0$, or, $C = n^2d$. Also $c - nd$ is then $= 0$, or $c = nd$; therefore

$$CS = \frac{n^2d - nxy - n^2y}{nd - ny} = n - \frac{xy}{d - y};$$

which is a general expression for the height of the centre of gravity for any finite altitude.

If we put $d - y = u$, then $\frac{d}{y} = \frac{1}{1 - \frac{u}{d}}$, and the $\log. \frac{d}{y} =$

$$\log. \frac{1}{1 - \frac{u}{d}} = \frac{u}{d} + \frac{u^2}{2d^2} + \frac{u^3}{3d^3} + \&c. = \frac{x}{n}, \text{ or}$$

$$x = n \times \left(\frac{u}{d} + \frac{u^2}{2d^2} + \frac{u^3}{3d^3} + \frac{u^4}{4d^4} + \&c. \right) \text{ and therefore}$$

$$\frac{xy}{d - y} = \frac{xy}{n} = n \times y \times \left(\frac{1}{d} + \frac{u}{2d^2} + \frac{u^2}{3d^3} + \frac{u^3}{4d^4} + \&c. \right)$$

$$\text{or } n - \frac{xy}{d - y} = n - n \times y \times \left(\frac{1}{d} + \frac{u}{2d^2} + \frac{u^2}{3d^3} + \frac{u^3}{4d^4} + \&c. \right).$$

Now when y is $= d$, or $u = 0$, all the terms of the series except the first will vanish, and we have $n - \frac{xy}{d - y} = n - n = 0$, as it ought to be.

Again, when y is $= 0$, or the altitude is infinite, u is $= d$, and we have

$$n - \frac{xy}{d-y} = n - 0 \times n \times (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.)$$

where it is evident that the product of the series by 0 is $=$ to 0,

and therefore when the altitude is infinite $n - \frac{xy}{d-y} = n$.

The problem may also be easily resolved by means of the logarithmic curve; for if the line CB (fig. 13, pl. 1.) denote the density of the air at the earth's surface, and the logarithmic curve BDE whose subtangent is n , be described; then the ordinates corresponding to the several altitudes will denote the densities of the air at those altitudes respectively (see Dalby's Course of Mathematics, vol. 2d, art. 444): and the centre of gravity of the cylindrical space, whose altitude is CD, will evidently be at the same distance from the surface of the earth, that the centre of gravity of the logarithmic space CDAB is from the base CB. Put $CB = d$, $x = DC$, and $y = DA$; then the fluxion of the space CDAB $= y\dot{x}$, and therefore the distance of

the centre of gravity of this space from the line CB is $= \frac{\int yx\dot{x}}{\int yx}$.

Also the equation of the curve is $x = n \times \log. \frac{d}{y}$, and therefore the process for finding the centre of gravity of the logarithmic space is precisely the same as before.

If S be the centre of gravity of the space CDAB, and SI, AV, be drawn perpendicular to CB, then we have $SI = n - \frac{x\dot{y}}{d-y} = n - \frac{CD \cdot DA}{VB} = \frac{n \cdot VB - CD \cdot DA}{VB}$. Or, because $n \cdot VB$ is $=$ to the area of the space CDAB (ibid. art. 443),

SI is $=$ to the $\frac{\text{area ABV}}{VB}$.

Now when CD is infinite, the points C, V, coincide, and the area of the whole logarithmic space, included between the asymptote and the curve, is $= n \cdot CB$, therefore in this case

SI is $= \frac{n \cdot CB}{CB} = n$, the same as determined above by expanding the logarithmic quantities.

SECOND SOLUTION, *by Mr. BAZLEY.*

If x denote the altitude from the surface, and y the density of the air at that altitude, then will hyp. log. $\frac{1}{y} = \frac{x}{p}$, where $p = 29725$ feet, the density at the surface being unity; this is well known. Now if the area of the base of the cylinder be taken for brevity's sake $= 1$, we shall have the fluxion of the quantity of matter $= y\dot{x} = -p\dot{y}$ by the equation above, and the fluent is py : but when $y = 1$ this fluent should vanish, therefore, by correction, the quantity of matter is $p - py$.

Again, to obtain the fluent of $yx\dot{x} = -px\dot{y}$; assume it $= -pxy + m$; whence $-px\dot{y} = -px\dot{y} - p\dot{y}x + \dot{m}$: therefore $m = p\dot{y}x = -p^2\dot{y}$, and $m = -p^2y$; consequently the fluent of $yx\dot{x}$ is $-pxy - p^2y$, which ought to vanish when $x = 0$, or $y = 1$; therefore, by correction, it is $p^2 - p^2y - pxy$. Hence the distance of the centre of gravity from the surface

$$= \frac{yx\dot{x}}{y\dot{x}} = \frac{p^2 - p^2y - pxy}{p - py} = p - \frac{xy}{1 - y}.$$

Mr. Toplis also sent a Solution to this Question.

XIII. QUESTION 133, *from the CAMB. UN. CAL.*

Required the nature of the curve along which a heavy body, descending by the force of gravity, will press upon the curve at any point, with a force proportional to the ordinate at that point?

FIRST SOLUTION, *by Mr. CUNLIFFE.*

The pressure of a body in motion against any point of a curve is manifestly composed of two parts, namely, the simple statical pressure and the pressure arising from the centrifugal force, or the force whereby the body endeavours to recede from the centre of curvature at any point: and upon the principle just cited it may be easily shewn that the quadrant of a circle, one of whose semi-diameters is perpendicular to the horizon will satisfy the question.

Let ABS (fig 14, pl. 1.) be the quadrant of a circle having the semi-diameter SB perpendicular to the horizon; and suppose a heavy body C, descends from rest at A, along the arch AC: draw CD perpendicular to AS, also draw the radius CS.

C.2

Put

Put $AS = SB = r$, $DC = y$, v = the velocity of the descending body per second at C , and $g = 16\frac{1}{2}$ feet.

Then by known principles $\frac{2gy}{r}$ will be as the simple statical pressure at C ; and $\frac{v^2}{r}$ will be as the pressure arising from the centrifugal force; therefore $\frac{2gy}{r} + \frac{v^2}{r}$ will be as the whole pressure at C ; and because the body descends from rest at A , $v = 2\sqrt{gy}$, or $v^2 = 4gy$; whence $\frac{2gy}{r} + \frac{v^2}{r} = \frac{2gy}{r} + \frac{4gy}{r} = \frac{6gy}{r}$; but g and r are both given; wherefore the whole pressure at C is as the ordinate $y = DC$.

COR. The whole pressure against any point C , of the quadrant, is thrice as great as the simple statical pressure at that point.

SECOND SOLUTION, by Mr. LOWRY.

Let ACB (fig. 14, pl. 1.) be the curve required, and C the position of the body when it has descended through the arc AC . Let mCn be a tangent to the curve at C , and CR , (drawn perpendicular to mCn) the radius of curvature at that point. Draw the abscissa AD parallel to the horizon, and the ordinate DC perpendicular to AD . Then the pressure of the body upon the curve will be in the direction RC , and will be composed of two parts, the one arising from the force of gravity, and the other from the actual velocity of the body. Put $AD = x$; $DC = y$, the space descended through, or the arc $AC = s$; the velocity of the body at $C = v$, and the measure of the accelerative force of gravity $= g$.

Then the force of gravity in the direction DC , is to its force in the direction SC , as radius to the cosine of the angle CSD or mCD , that is, as $y : x$; and therefore the pressure caused by gravity is $= g \times \frac{x}{y}$. Again since the velocity of the body in the curve at C , is the same as in the circle whose radius is CR , it is evident that the pressure arising from the velocity will be the same as the centrifugal force in the circle of curvature at C , that is $= \frac{v^2}{CR}$. Put since the body begins to descend

ascend from A, v^2 is $= 2gy$; therefore the pressure arising from the velocity is $= \frac{2gv}{CR}$, and the whole pressure upon the curve $= g \times \left(\frac{\dot{x}}{s} + \frac{2y}{CR} \right)$. But, by the question, the pressure is proportional to the ordinate DC, therefore $g \left(\frac{\dot{x}}{s} + \frac{2y}{CR} \right) = \frac{m}{n} \cdot y$, where $\frac{m}{n}$ denotes a constant ratio.

Now when $\dot{s}$ is constant, the radius of curvature is $= \frac{\ddot{sy}}{x}$, and putting this for CR in the equation above, we have

$$\frac{\dot{x}}{s} + \frac{2\dot{x}}{\dot{sy}} = \frac{m}{ng} y;$$

multiply both sides of this equation by $\frac{\ddot{sy}}{\sqrt{y}}$,

$$\text{then } \frac{\dot{y}}{\sqrt{y}} + 2\dot{x} \sqrt{y} = \frac{m}{ng} \cdot \ddot{sy} \sqrt{y},$$

take the fluents and then

$$2x \sqrt{y} = \frac{2m}{3ng} \cdot \dot{sy}^{\frac{3}{2}}, \text{ or } \dot{x} = \frac{m}{3ng} \cdot \dot{sy}.$$

To exterminate $\dot{s}$, we have $\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2)}$,

$$\text{therefore } \dot{x} = \frac{m}{3ng} \cdot \sqrt{(\dot{x}^2 + \dot{y}^2)} \cdot y,$$

squaring both sides and reducing we have

$$\dot{x} = \frac{m\dot{y}}{\sqrt{9n^2g^2 - m^2y^2}}, \text{ and taking the fluents}$$

$$\dot{x} = - \sqrt{\left(\frac{9n^2g^2}{m^2} - y^2 \right)}, \text{ which gives } x = \frac{3ng}{m} \text{ when } y \text{ is } 0,$$

but x , should then be also $= 0$; therefore the correct fluent is

$$x = \frac{3gn}{m} - \sqrt{\left(\frac{9n^2g^2}{m^2} - y^2 \right)}; \text{ and } \frac{6ng}{m} \cdot x - x^2 \text{ is } y^2;$$

which is an equation to a circle whose radius is $\frac{3ng}{m}$.

Solutions were also received from Messrs. Bazley and Toplis.

XIV. QUESTION 134, *from the CAMB. UN. CAL.*

Let the curve APp be generated by taking the ordinate MP always equal to the difference of the chord AC, and the versed sine AM, of the circular arc AC. It is required to determine the greatest ordinate of the curve, and also its area?

FIRST SOLUTION, *by the Rev. Mr. TOPLIS.*

Fig. 15, pl. 1. Put $AM = x$, $PM = y$, and the diameter $AB = a$; then, by the property of the circle, $AC = \sqrt{(AM \cdot AB)} = \sqrt{(ax)}$, and therefore $y = \sqrt{(ax)} - x$ the equation of the curve.

To find when y is a maximum we have $\frac{\dot{x}\sqrt{a}}{2\sqrt{x}} - \dot{x} = 0$,
or $\sqrt{a} = 2\sqrt{x}$, and consequently $x = \frac{1}{4}a$ and $y = \sqrt{\frac{a^2}{4}} - \frac{1}{4}a$
 $= \frac{1}{2}a - \frac{1}{4}a = \frac{1}{4}a$.

Again, to find the area we have $y\dot{x} = \dot{x}\sqrt{(ax)} - x\dot{x}$, and
the fluent $= \frac{2}{3}\sqrt{(ax^3)} - \frac{x^2}{2} =$ (when $x = a$) to $\frac{2}{3}a^2 - \frac{a^2}{2}$
 $= \frac{1}{6}a^2$ the area required.

The whole might however have been determined without fluxions, for the curve is only a portion of the common parabola, as is evident from the equation obtained above, for $y + x =$ to $\sqrt{(ax)}$, or $(y + x)^2 = ax = x\sqrt{2} \times a\sqrt{\frac{1}{2}}$, which is an equation to a parabola whose ordinate is $y + x$, abscissa $x\sqrt{2}$ and the parameter $a\sqrt{\frac{1}{2}}$.

SECOND SOLUTION, *by Mr. BAZLEY.*

Fig. 15, pl. 1. Suppose O the centre of the circle, AMB the diameter, and IR another diameter perpendicular thereto; through R draw ARK, and with the parameter AR and vertex A describe a parabola to the diameter AK having its ordinates parallel to IR; the portion APpB of this parabola is the locus required.

Let CM produced, meet AR in G, and join RB. Then, since $OR = OA$, AM is to MG, and by the property of the parabola and construction $RA \times AG = GP^2$; but by the similar triangles AMG, ARB, $RA \times AG = BA \times AM = AC^2$; there-
fore

fore $GP^2 = AC^2$ or $GP = AC$: whence $AC - AM = GP - GM = MP$, which was to be proved.

Now from O draw Op parallel to AK to meet the curve in p; draw pT parallel to AB meeting KA produced in T; also draw pmg parallel to PMG meeting AO in m and AK in g. Then since pO is a diameter of the parabola and O the middle of AB, therefore AB is an ordinate to pO, and pT, parallel to AB, is a tangent at p, and consequently all the ordinates except pm fall below pT; therefore pm is the greatest ordinate; moreover, by the property of the tangent, $gA = AT$, therefore $Am = \frac{1}{2}pT = \frac{1}{2}AO$.

Finally for the area: it is well known that the area GAP is $= \frac{2}{3}GP \times AM = \frac{2}{3}AC \times AM$, therefore the area APM $= \frac{2}{3}AC \times AM - \frac{1}{2}AM \times MG = \frac{2}{3}AC \times AM - \frac{1}{2}AM^2$, which is the general area. Let BK parallel to IR meet AK in K; then it is plain that when M comes to B, the whole area APpB is $\frac{2}{3}KB \times BA - \frac{1}{2}AB \times BK = \frac{2}{3}AB^2 - \frac{1}{2}AB^2 = \frac{1}{3}AB^2$.

Mr. Dawes and J. W. of Leeds, also answered it.

XV. QUESTION 135, from the CAMB. UN. CAL.

Determine the apparent magnitude of a straight rectilinear object placed at a given depth, parallel to a surface of water: the eye being situated at any point in the plane passing through the object perpendicular to the surface?

FIRST SOLUTION, by AMICUS.

Let AB (fig. 16, pl. 1.) represent the straight rectilinear object placed parallel to the surface of the water RQ, and P the position of the eye in the plane ABP perpendicular to RQ, and let CLD be the curvilinear image of the object as it appears to the eye at P. See Emerson's Optics, Prop. 29. B. 3. Then the extremities A, B, of the object will appear to be elevated to the points C, D, in the perpendiculars AE, BF. Draw CP, DP, meeting EF in G and H; Join A, G; B, H; A, P; B, P; and draw GN perpendicular to FE, then CGN is equal to the angle of incidence, and AGN is the angle of refraction; therefore by the principles of optics, when the medium in which the object is immersed is water, $\sin CGN : \sin AGN :: 4 : 3 :: m : n$, or, since the angle CGN = GCE and the angle AGN = CAG, we have by trigonometry

$$AG : CG :: \sin. GCE : \sin. CAG :: m : n.$$

Put $PK = a$, $EK = b$, $AE = c$, and $EG = x$, then AG^2

$AG^2 = AE^2 + EG^2 = c^2 + x^2$, and by similar triangles
 $b - x : a :: x : \frac{ax}{b - x} = CE$; hence $CG^2 = CE^2 + EG^2 = \frac{a^2 x^2}{(b - x)^2} + x^2$,

and $c^2 + x^2 : \frac{a^2 x^2}{(b - x)^2} + x^2 :: m^2 : n^2$. By multiplying means and extremes, and reducing, we obtain $(m^2 - n^2) x^4 + 2(m^2 - n^2) b^3 + \{n^2 a^2 + b^2\} - m^2(c^2 + b^2)\} x^2 + 2m^2 b c^2 x - n^2 c^2 b^2 = 0$; from which equation the value of x may be found, when the given quantities are expressed in numbers, and then the point G and the distance GK become known. Also, from the same equation, we may determine the distance HF , by substituting the value of KF instead of b (EK), and then the point K and the distance KH become known.

We have then only to determine the values of the angles GPH , APB ; and by trigonometry
 $PK : KG :: 1 \text{ (rad.)} : KG \div PK = \text{tang. of the } \angle KPG$, and
 $PK : KH :: 1 \text{ (rad.)} : KH \div PK = \text{tang. of the } \angle KPH$;
 then the sum of the angles corresponding to these tangents in the tables, will be the angle GPH . And in a similar way may the angle APB be calculated, and from the values of these angles we can determine the comparative magnitudes of the object and its image.

SECOND SOLUTION, by Mr. GEORGE FRANCIS, OPTICIAN,
No. 5, Old Cavendish Street, Cavendish Square, London.

Let OB (fig. 17, pl. 1.) be the object, VE and VC the distances of the eye and object from the surface of the water, and let EA, Ea , be rays incident at the points A, a , so as to be refracted to the given points B, O . Draw BP, Op , perpendicular to Aa , and produce EA, Ea , to meet OB produced in D, d ; then OEB will be the visual angle when the object OB is not immersed in the fluid and DEd the visual angle when it is immersed. Let $I : R :: 1\frac{1}{2} : 1$, then by prop. 11, of Dr. Gregory's Optics, the times of describing the rays EA, AB , will be expressed by $R \cdot EA$ and $I \cdot AB$, and their sum $R \cdot AE + I \cdot AB$ a minimum. Put $VE = d, BC = VP = a, VC = BP = c$, and $PA = x$; then $VA = a - x, EA = \sqrt{(d^2 + a^2 - 2ax + x^2)}$, $BA = \sqrt{(c^2 + x^2)}$, Eu. 47. I; and

$$R \cdot EA + I \cdot AB = R \sqrt{(d^2 + a^2 - 2ax + x^2)} + I \sqrt{(c^2 + x^2)},$$

$$\text{taking the fluxion, } \frac{I \times x\dot{x}}{\sqrt{(c^2 + x^2)}} + \frac{R \times (-a\dot{x} + x\dot{x})}{\sqrt{(d^2 + a^2 - 2ax + x^2)}} = 0,$$

and

and reducing, we find $(T^2 - R^2)x^4 + (2aR^2 - 2aI^2)x^3 + (I^2d^2 + I^2a^2 - R^2a^2 - R^2c^2)x^2 - 2R^2ac^2x \pm R^2a^2c^2$.

From this equation, the value of x , or the distance AP may be found, and also the value of ap , by substituting pV instead of PV . Then, in the right angled triangles aVE , AVE , two sides will be given in each, to determine the angles VEA , VEa , and then the sum of these angles will be the visual angle DED .

XVI. QUESTION 136, from the CAMB. UN. CAL.

Let the roots of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, be $a, b, c, \&c.$ and those of the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} - \&c. = 0$, be $\alpha, \beta, \gamma, \&c.$ then, if when $\alpha, \beta, \gamma, \&c.$ are successively substituted in the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, the results are $P, Q, R, \&c.$ and when $a, b, c, \&c.$ are substituted in the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} - \&c. = 0$, the results are $p, q, r, \&c.$ it will be as $P \times Q \times R, \&c. : p \times q \times r \&c. :: 1 : n^n$. Required the demonstration?

SOLUTION, by LIMENUS.

The equation of n dimensions $= (x-a)(x-b)(x-c) \dots$ being denoted by P , and that of $n-1$ dimensions $= n(x-a)(x-\beta)(x-\gamma) \dots$ by Q , the former expression when the indeterminate x is of the values $\alpha, \beta, \gamma, \dots$ which would make the latter vanish, is equal to $(\alpha-a)(\alpha-b)(\alpha-c) \dots, (\beta-a)(\beta-b)(\beta-c) \dots, \&c.$ respectively, and and their product is therefore equal to the product of all the excesses of $\alpha, \beta, \gamma, \dots$ above $a, b, c, \dots$ taken two and two, in any order. Again, when P vanishes, or when x is equal to $a, b, c, \dots$ respectively, the expression Q is of the several values $n(a-a)(a-\beta) \dots, n(b-a)(b-\beta) \dots, n(c-a)(c-\beta) \dots, \&c.$ and the product of these is equal to n^n multiplied by the product of the excesses of $a, b, c, \dots$ above $\alpha, \beta, \dots$ taken two and two. As then these excesses are in each case the same with contrary signs, and their number $= n(n-1)$ is even, the above values of P multiplied together are to those of Q multiplied together in the ratio of 1 to n^n .

XVII. QUESTION 137, by Mr. HENRY LIGHTBOWN.

Suppose a heavy flexible chain of a given length, having its ends fastened together, to be thrown over the vertex of a given square pyramid whose surface is perfectly polished; what portion of the surface will be included between the vertex and the chain, when it has descended as low as it can get?

SOLUTION, by Mr. CUNLIFFE.

When the chain has descended as low as it can get, it is manifest that the four parts thereof lying upon the four equal sides of the surface of the pyramid will be equal to each other. Let the isosceles triangle ACB (fig. 18, pl. 1.) represent one of the equal sides of the surface of the pyramid; DVE the catenaria, or part of the chain lying thereon; then when the chain has descended as low as it can get, CE and CD will be equal to each other, and both perpendicular to the curve at the points D and E. This will be pretty clear, when it is considered that the force of tension at any point acts in the direction of the curve at that point, and at E and D the tension acting perpendicular to CE, CD can have no tendency whatever to move the chain either up or down the lines CE, CD; it must therefore be in equilibrio in that circumstance.

These things being premised, join DE, and draw CF perpendicular to AB, cutting DE in R and the catenaria in V. Draw re indefinitely near and parallel to RE meeting the curve in e, also draw en parallel to CF. Put $VE = l = \frac{1}{n}$ th part of the given length of the chain, $VR = x$, $RE = y$; and let a denote a portion of the chain whose weight is equal to the tension at V; then $ne = Rr = -\dot{x}$, and $nE = \dot{y}$; also let the ratio of RE to RC, which is given, be denoted by that of 1 to π . Then per similar triangles enE, CRE, $nE : ne :: RE : RC :: 1 : \pi$, whence $\pi \times nE = ne$, that is $n\dot{y} = -\dot{x}$. Now by known properties of the curve $y = a \times \text{h. l. } \frac{l + \sqrt{(l^2 + a^2)}}{a}$, and $x = \sqrt{(l^2 + a^2)} - a$, taking the fluxions of these expressions, supposing a variable, $\dot{y} = \dot{a} \times \text{h. l. } \frac{l + \sqrt{(l^2 + a^2)}}{a} - \frac{l\dot{a}}{\sqrt{(l^2 + a^2)}}$, and $\dot{x} = \frac{a\dot{a}}{\sqrt{(l^2 + a^2)}} - \dot{a}$, or $-\dot{x} = \dot{a} - \frac{a\dot{a}}{\sqrt{(l^2 + a^2)}}$, which being written in the equation $n\dot{y} = -\dot{x}$ it becomes

$\pi n\dot{a}$

$$na \times \text{h. l. } \frac{l + \sqrt{l^2 + a^2}}{a} - \frac{nla}{\sqrt{l^2 + a^2}} = a - \frac{aa}{\sqrt{l^2 + a^2}},$$

$$\text{dividing by } na, \&c \text{ h. l. } \frac{l + \sqrt{l^2 + a^2}}{a} = \frac{l + \frac{1}{n} \sqrt{l^2 + a^2} - \frac{a}{n}}{\sqrt{l^2 + a^2}},$$

from whence the relation of a and l may be determined by the method of trial and error, or by other methods of approximation.

Now suppose the relation of a and l has been found and is expressed by the equation $a = ml$, then $RE = a \times \text{h. l. } \frac{l + \sqrt{l^2 + a^2}}{a} = ml \times \text{h. l. } \frac{1 + \sqrt{1 + m^2}}{m}$, whence

$$RC = n \times RE = nml \times \text{h. l. } \frac{1 + \sqrt{1 + m^2}}{m}, \text{ and}$$

$$RV = \sqrt{l^2 + a^2} - a = l \times \{ \sqrt{1 + m^2} - m \}.$$

Then by means of these values of RE , RC , and RV , the areas of both the triangles DCE and the catenary DVE may be found, and their sum is the space $CDVE$, which will be $\frac{1}{4}$ th part of the portion of the surface of the pyramid included between the vertex and the chain.

XVIII. QUESTION 138, by Mr. THOMAS BAZLEY, Bolton.

Suppose a perfectly flexible line or chain of a given length BNR to have one end B fastened to an immoveable tack, and a ring R at the other end put upon a smooth inflexible rod RO , given by position in the same vertical plane with the point B : it is required to determine that point upon the rod at which the ring will rest?

SOLUTION, by Mr. SENEX.

Draw BO (fig. 19, pl. 1.) parallel to the horizon, meeting RO in O , and imagine the catenary BNR to be continued till it meets BO in A ; also draw the axis MN and the ordinate RS . When the chain is in equilibrio the curve of the catenary will be perpendicular to RO at R ; for in that case all the force at R is acting at right angles to RO , and can therefore have no tendency whatever to move the chain either up or down the rod RO , consequently, the chain must rest in equilibrio, in that circumstance. The preceding matters having been laid down,

D 2

draw

draw rs indefinitely near and parallel to RS , also draw rnq parallel to MN meeting RS in n , and RO in q . Then because rRq is a right angle, the triangles rnR , Rnq will be similar, and $rn : nR :: nR : nq$, a given ratio, because the position of the line RO is given: let this ratio be denoted by that of n to 1. Put the length of the chain $BNR = l$, $NS = x$, $SR = y$, $NM = z$, and let a denote a portion of the chain whose weight is equal to the tension at N ; then $rn = \dot{x}$ and $nR = \dot{y}$; also put $OB = b$, and by the property of the curve, the length of the curve $RN = \sqrt{(2ax + x^2)}$, and the length of the curve $BN = \sqrt{(2az + z^2)}$; therefore $BN + RN = \sqrt{(2az + z^2)} + \sqrt{(2ax + x^2)} = l$. Again, by the property of the curve and what has been deduced $\dot{x} : \dot{y} :: \sqrt{(2ax + x^2)} : a :: n : 1$, whence $\sqrt{(2ax + x^2)} = na$ and $x^2 + 2ax = n^2 a^2$, completing the square $x^2 + 2ax + a^2 = a^2 \times (1 + n^2)$, taking the roots $x + a = a\sqrt{(1 + n^2)}$, and $x = a \times \{ \sqrt{(1 + n^2)} - a \}$; also $\frac{a + x + \sqrt{(2ax + x^2)}}{a} = \sqrt{(1 + n^2)} + n$.

Again from the equations $\sqrt{(2az + z^2)} + \sqrt{(2ax + x^2)} = l$, and $\sqrt{(2ax + x^2)} = na$, we get $\sqrt{(2az + z^2)} = l - na$; whence $z = \sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} - a$, $z - x = \sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} - \sqrt{a^2(1 + n^2)}$ and $\frac{a + z + \sqrt{(2az + z^2)}}{a} = \frac{\sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} + l - na}{a}$.

Now draw Op , Bb parallel to MN meeting RS in p and b ; then we shall have

$Op = MS = MN - NS = z - x = \sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} - \sqrt{a^2(1 + n^2)}$. Again, by known properties of the curve,

$RS = a \times \text{h. l. } \frac{a + x + \sqrt{(2ax + x^2)}}{a} = a \times \text{h. l. } \{ \sqrt{(1 + n^2)} + n \}$

$MB = a \times \text{h. l. } \frac{a + z + \sqrt{(2az + z^2)}}{a} = a \times \text{h. l. } \frac{\sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} + l - na}{a}$

whence we shall have

$Rp = RS + Sb - pb = RS + MB - OB$

$= a \times \text{h. l. } \{ \sqrt{(1 + n^2)} + n \} + a \times \text{h. l. } \frac{\sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} + l - na}{a} - b$.

Furthermore we have $Rp : Op :: n : 1$, whence $Rp = n \times Op$, that is, by means of what has been deduced

$a \times \text{h. l. } \{ \sqrt{(1 + n^2)} + n \} + a \times \text{h. l. } \frac{\sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} + l - na}{a} = b$.

$= n \sqrt{(l^2 - 2lna + n^2 a^2 + a^2)} - na \sqrt{(1 + n^2)}$, or

$a \times$

$a \times h. l. \{ (1 + n^2) + n \} + a \times h. l. \frac{\sqrt{(l^2 - 2lna + n^2a^2 + a^2)} + l - na}{a}$
 $= n \sqrt{(l^2 - 2lna + n^2a^2 + a^2)} - na \sqrt{(1 + n^2)} + b$, from which
 expression a may be found by the method of trial and error, and
 hence all the rest becomes known.

XIX. QUESTION 139, by Mr. LOWRY.

In a given trapezium to inscribe another, such, that three of its sides may pass through given points and the remaining side be of a given length?

SOLUTION, by the Proposer.

Let $ABGD$ (fig. 20, pl. 1.) be the given trapezium, and $PQRS$ the inscribed one, having three of its sides passing through the given points E, F, G , and the remaining side PS of a given length. Draw EI parallel to AB , meeting BG in I , and let the line joining the points I, F , meet GD in i ; join i, C , and draw Bi parallel to Ii meeting GD in a . Let the line AE meet BG in H , and draw HF meeting GD in h and Bi produced in K ; then it is evident that the points I, H, i, h, a , are given, and therefore the point K will be given, and the line KB will be given in magnitude and position.

Now by reason of the parallels,

$$EI : AB :: HI : HK :: IF : BK,$$

$$\text{and } EI : BP :: IQ : BQ :: IF : BL,$$

$$\text{therefore } AB : BP :: BK : BL,$$

$$\text{and convertendo } AB : AP :: BK : KL,$$

$$\text{and permutando } AB : BK :: AP : KL;$$

therefore, because AB and BK are given lines, AP has to KL a given ratio.

Again, draw Ch , and through the point a draw akl parallel to iG meeting Ch produced in k and RS in b ; then, because C, i, h, a , are given points, it is evident that k will also be a given point, and the line ka given in magnitude and position.

$$\text{And by the parallels, } Fi : aK :: ih : ah :: iC : ak,$$

$$\text{and } Fi : aL :: iR : aR :: iC : al,$$

$$\text{therefore } aK : aL :: ak : al,$$

$$\text{and convertendo } aK : LK :: ak : kl,$$

$$\text{and permutando } aK : ak :: LK : kl,$$

therefore, since aK and ak are given lines, LK has to kl a given ratio

ratio. Let this ratio be compounded with the ratio of AP to KL, and we have

$AB : AK :: BK : ak :: AP : kl$, a given ratio.

On Ck take CV to Ck in this ratio, and draw VT parallel to kl meeting RS in T; then V will be a given point, and the line VT will be given by position. Then by the parallels and construction $CV : Ck :: VT : kl :: AP : kl$; therefore VT will be equal to AP, and the problem is now reduced to the following one, *viz.*

To draw CTS to meet VT in T and AD in S, so that taking AP equal to VT, the line PS may be of the given magnitude. And this problem admits of distinct cases, according as the line VT is parallel, or not parallel, to the side AD.

CASE I. When VT is parallel to AD. Fig. 21, pl. 1.

Through C draw NO parallel to AP, meeting AD in N, and VT in O, and take AI = to VO; then because $AP = VT$, PI is = TO, and by similar triangles $NS : TO = PI :: NC : CO$. Draw SY, IY, parallel to AI and AN, and let NY be drawn to meet AI produced in W; then SY is = PI, and YI = SP, and by the above proportion $NS : SY :: (AN : AW) :: NC : CO$ a given ratio, because NC and CO are given lines; hence we have this construction:

Take AN to AW in the given ratio of NC to CO, join NW, and from I apply IY = the given line SP; draw SY parallel to AW and the point S will be determined.

CASE II. When VT is not parallel to AD. Fig. 22, pl. 1.

Let VT be produced to meet AD in L, and draw CN parallel to VN and CO parallel to AD. Take AI = to VO, and draw IY parallel to PS and SY parallel to AI. Join NY and draw NZ parallel to SY. Then because $AP = VT$, TO is = PI = SY, and by similar triangles $NS : NC :: OC : TO = SY$, therefore $NS \cdot SY = CN \cdot OC =$ to a given rectangle, because CN and OC are given lines; hence the locus of the point Y is a hyperbola whose centre is N and asymptotes AN and ZN; and if this hyperbola be described, and IY = the given line PS, be applied from I to the curve at Y, then YS being drawn parallel to IA, its intersection with AD will determine the point S.

XX. QUESTION 140, by Mr. W. SIMPSON, Bolton.

$$\begin{aligned} n \times \left\{ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c. \text{ ad infinitum} \right\} \\ = \frac{x^n-1}{x^n} + \frac{1}{2} \left(\frac{x^n-1}{x^n} \right)^2 + \frac{1}{3} \left(\frac{x^n-1}{x^n} \right)^3 + \frac{1}{4} \left(\frac{x^n-1}{x^n} \right)^4 + \frac{1}{5} \left(\frac{x^n-1}{x^n} \right)^5 + \&c. \\ \text{ad infinitum.} \end{aligned}$$

Required the demonstration?

SOLUTION,

SOLUTION, by the Rev. J. TOPLIS, *Arnold, Notts.*

It is well known that h. l. $(1+r) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.$ write $x - 1$ in the place of x and the expression will become h. l. $(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c.$ and by the nature of logarithms h. l. $(x^n) = n \times$ h. l. $(x) = n \times \left\{ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c. \text{ ad infinitum} \right\}$. Again it is well known that h. l. $\left(\frac{1}{1-x}\right) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \&c.$ in which expression writing $\frac{x^n-1}{x^n}$ in the place of x we shall have

$$\begin{aligned} \text{h. l. } (x^n) &= \left(\frac{x^n-1}{x^n}\right) + \frac{1}{2}\left(\frac{x^n-1}{x^n}\right)^2 + \frac{1}{3}\left(\frac{x^n-1}{x^n}\right)^3 + \frac{1}{4}\left(\frac{x^n-1}{x^n}\right)^4 + \&c. \\ \text{ad infinitum.} \end{aligned}$$

Therefore $n \times \left\{ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c. \text{ ad infinitum.} \right\} = \frac{x^n-1}{x^n} + \frac{1}{2}\left(\frac{x^n-1}{x^n}\right)^2 + \frac{1}{3}\left(\frac{x^n-1}{x^n}\right)^3 + \frac{1}{4}\left(\frac{x^n-1}{x^n}\right)^4 + \&c. \text{ ad infinitum.}$

Q. E. D.

XXI. QUESTION 141, by Mr. CUNLIFFE.

It is required to find three whole numbers, such, that the sum of every two may be a cube; and also that the sum of all the three may be a cube?

SOLUTION, by J. W. of Leeds.

Let x, y , and z , denote the numbers; and by the question we shall have

$$\begin{aligned} x + y + z &= a^3 \\ x + y &= b^3 \\ x + z &= c^3. \end{aligned}$$

The difference between twice the first equation and the sum of the other two is $y + z = 2a^3 - b^3 - c^3$, which is also to be a cube by the question.

Put

$$\begin{aligned} \text{Put } a &= b + n, \text{ and then } 2a^2 - b^2 - c^2 = 2(b+n)^2 - b^2 - c^2, \\ &= b^2 + 6b^2n + 6bn^2 + 2n^2 - c^2 = \text{a cube} = (b + 2n)^2, \\ &= b^2 + 6b^2n + 12bn^2 + 8n^2, \text{ which gives } b = \frac{-(c^2 + 6n^2)}{6n^2}, \end{aligned}$$

$$a = b + n = \frac{-c^2}{6n^2}, \text{ and } c = \frac{6cn^2}{6n^2}; \text{ writing } p \text{ for } -c, q \text{ for } n$$

and rejecting the common denominator $6n^2$, there will be had, $a = p^2$, $b = p^2 - 6q^2$ and $c = -6pq^2$. From these conclusions a general solution to the question might be obtained, but the calculation would be rather oporose and the resulting expressions complex; and besides I have not discovered that a general solution founded upon what is here deduced has any advantage in determining numbers moderately small that will answer the question. I shall therefore proceed with the solution in a manner something less general. For this purpose take $p = 3$, and $q = 1$, then $a = 27$, $b = 21$, and $c = -18$; and as each of these is divisible by 3 we may take $a = 9$, $b = 7$, and $c = -6$; but c is negative, therefore in order to find positive numbers that will answer, put $a = rv + 9$, $b = sv + 7$ and $c = tv - 6$; then $2a^2 - b^2 - c^2 = 2(rv + 9)^2 - (sv + 7)^2 - (tv - 6)^2$, $= v^2(2r^2 - s^2 - t^2) + 3v^2(18r^2 - 7s^2 + 6t^2) + 3v(162r - 49s - 36t) + 1331 = \text{a cube} = (gv + 11)^2 = g^2v^2 + 33g^2v + 363gv + 1331$; put $3v(162r - 49s - 36t) = 363gv$ in order that two of the latter terms on each side of the equation may destroy each other; whence $g = \frac{162r - 49s - 36t}{121}$; and

from the equality of the remaining terms, viz.

$$v^2(2r^2 - s^2 - t^2) + 3v^2(18r^2 - 7s^2 + 6t^2) = g^2v^2 + 33g^2v^2,$$

$$v = \frac{33g^2 - 3(18r^2 - 7s^2 + 6t^2)}{2r^2 - s^2 - t^2 - g^2}, \text{ where } r, s, \text{ and } t \text{ may be}$$

taken at pleasure.

$$\text{Take } r, s, \text{ and } t, \text{ each} = 1, \text{ then } g = \frac{7}{11} \text{ and } v = \left(\frac{33 \times 49}{121} - 51 \right)$$

$$\div \left(\frac{-7}{11} \right)^2 = \left(51 - \frac{3 \times 49}{343} \right) \div \left(\frac{7}{11} \right)^2 = \frac{50094}{343}, \text{ whence}$$

$$a = rv + 9 = \frac{53181}{343}, b = sv + 7 = \frac{52495}{343}, c = tv - 6 = \frac{48036}{343},$$

and rejecting the common denominator, 343, we may take $a = 53181$, $b = 52495$, and $c = 48036$.

Whence

Whence $a^3 = 150407501928741$

$b^3 = 144661785187375$

$c^3 = 110841018670656$

$x = b^3 + c^3 - a^3 = 105095301929290$

$y = a^3 - c^3 = 39566483258085$

$z = a^3 - b^3 = 5745716741366$

which are three numbers that will answer.

And thus nearly the question was answered by Mr. Cunliffe, the Proposer.

XXII. QUESTION 142, by Mr. CUNLIFFE.

P is a given point in the diameter AB produced of a given circle, let any line PCD be drawn cutting the circle in C and D, and let AC, BD be drawn intersecting in I: now, if in BD produced there is continually taken $DR = DI$, required the equation and quadrature of the curve which is the locus of R?

FIRST SOLUTION, by Mr. LOWRY.

Fig. 23, pl. 1. Let PF be a tangent to the circle at F and draw FG perpendicular to the diameter AB. Then AB is divided into given segments AE, BE, and the line FG is given by position and is the locus of the intersection I of the chords AC, BD (see the solutions to the prize question in the 12th number of the Old Series of the Repository). Hence, we have to determine the locus of the point R when the distance RD is constantly equal to the distance DI intercepted between the circle and the straight line FG given by position. On AB produced take $AQ = AE$ and $BL = EB$; draw OLT perpendicular to AL and join AD. Put $AB = d$, $BE = c$, $BN = x$, and $RN = y$; then $BR = \sqrt{x^2 + y^2}$, and by similar triangles,

$BR : BN :: AB : BD = \frac{dx}{\sqrt{x^2 + y^2}}$; again, by similar triangles,

$AB : BD :: IB : BE$, and by permutation and composition, $IB : IB + BR :: BE : BE + BN$, or, $AB : IB + BR :: BD : BE + BN$. But since $RD = DI$, $IB + BR = 2BI + 2DI = 2DB$;

therefore $AB : 2DB :: DB : BE + BN = \frac{2BD^2}{d}$, or $c + x = \frac{2dx^2}{x^2 + y^2}$;

this reduced gives $y = x \sqrt{\frac{2d - (c + x)}{c + x}}$ for the equation of the curve.

Now, in this equation, when x is either 0, or equal to $2d - c$, that is equal to BQ, then y is 0, which shews that the curve passes through the points B, Q. Also when x is equal to c , then $y = c \sqrt{\frac{2d - 2c}{2c}} = \sqrt{dc - c^2} = EF$, and the curve

passes through the point F. In like manner, if the ordinate y be taken on the other side of BQ, then the curve will pass through the points B, G, and B will be a double point in the curve. Moreover, if x be taken from B towards

P, then y is $= x \sqrt{\frac{2d - (c - x)}{c - x}}$, and when x is $=$ to c (or to BL), then y is infinite, which shews that the line OLT is an asymptote to the curve.

To find the area, put $z = 2d - (c + x)$, that is $=$ to QN,

then y is $= \{2d - (c + z)\} \sqrt{\frac{z}{2d - z}}$, and the fluxion of the

area QRN $= y\dot{z} = \dot{z} \{2d - (c + z)\} \sqrt{\frac{z}{2d - z}} = \frac{2\dot{z}(2d - z)}{\sqrt{(2dz - z^2)}}$

$-\frac{cz\dot{z}}{\sqrt{(2dz - z^2)}} = \dot{z} \sqrt{(2dz - z^2)} - \frac{cz\dot{z}}{\sqrt{(2dz - z^2)}}$. On the

diameter QL (which is equal to $2AB = 2d$) describe a circle LMQ, and let the ordinate NR meet the circle in M; then the fluent of $\dot{z} \sqrt{(2dz - z^2)}$ is the circular segment QMN;

and the fluent of $\frac{cz\dot{z}}{\sqrt{(2dz - z^2)}}$ is the arc QM $-\sqrt{(2dz - z^2)}$

$=$ the arc QM $- MN$; therefore the area QRN is $=$ to the segment QMN $-(\text{arc QM} - MN) \times c$. And the whole area BFQGB is $=$ to the segment HQh $-(\text{arc HQh} - Hh) \times c$, Hh being drawn perpendicular to AB.

Again the fluxion of the area Bnr

is $= \frac{cz\dot{z}}{\sqrt{(2dz - z^2)}} - \frac{\dot{z}(2d - z)}{\sqrt{(2dz - z^2)}}$, and the fluent $=$ (arc

Qm $- mn) \times c$ $-$ segment Qmn $- C$. Now when z is $=$ QB, or $2d - c$, the area Bnr is 0; but the fluent is then $=$ to the segment QhB $-(\text{arc Qh} - hB) = C$; therefore the correct fluent for the area is (arc hm $- ah) \times c$ $-$ area hBnm. And the area of the whole space contained between the curve and the asymptote indefinitely produced is $=$ to (arc HLh $- Hh) \times c$ $-$ segment HLh.

SECOND

SECOND SOLUTION, by Mr. CUNLIFFE, the Proposer.

Draw the tangents PT , PT' (fig. 24, pl. 1.) and join TT' cutting the diameter AB in Q ; then the right line TT' is the locus of the intersection of AC and BD , this is pretty generally known. In AB produced take $AV = AQ$ and V will be a point in the curve as appears from its generation. Draw RS , DG , perpendicular to BA and join AD ; then we shall have $QV = 2QA$, and $QS = 2QG$, whence $SV = QV - QS = 2QA - 2QG = 2 \times (QA - QG) = 2AG$, or $AG = \frac{1}{2}SV$. Put $BA = a$, $AQ = AV = b$, $VS = x$, and $SR = y$, then $AG = \frac{x}{2}$, $BG = a - \frac{x}{2}$, $BS = a + b - x$;

and per similar triangles BSR , BDA ;

$BD^2 : DA^2 :: BS^2 : SR^2$ and $BD^2 : DA^2 :: BG : GA$, therefore $BG : GA :: BS^2 : SR^2$; whence $BG \times SR^2 = GA \times BS^2$;

that is $(a - \frac{x}{2})y^2 = \frac{x}{2}(a + b - x)^2$ which gives $y^2 = \frac{x(a + b - x)^2}{2a - x}$,

which is the equation of the curve.

The following properties are pretty obviously suggested by the equation.

When $x = b + a = VA + AB = VB$, then $y = 0$, which shews that the curve cuts the axis again in B , the point B being what is commonly called the punctum duplex.

When $x = 2b = VQ$, then $y^2 = 2b(a - b)^2 \div 2(a - b) = b(a - b) = AQ \times BQ = QT^2$, which shews that the curve passes through T and T' , and this is also manifest from the generation of the curve.

When $x = 2a$, then y is infinite; therefore the curve has an asymptote perpendicular to VB at the distance $2AB$ from V .

Again we have $y\dot{x} = \frac{(a + b)x\dot{x} - x^2\dot{x}}{\sqrt{(2ax - x^2)}}$ for the fluxion of the area. Assume $Ax\sqrt{(2ax - x^2)} + B\sqrt{(2ax - x^2)} + C \times \frac{\dot{x}}{\sqrt{(2ax - x^2)}}$
 $= \int \frac{(a + b)x\dot{x} - x^2\dot{x}}{\sqrt{(2ax - x^2)}}$; taking the fluxions $A\dot{x}\sqrt{(2ax - x^2)}$
 $+ \frac{Ax(a\dot{x} - x\dot{x})}{\sqrt{(2ax - x^2)}} + \frac{B(a\dot{x} - x\dot{x})}{\sqrt{(2ax - x^2)}} + \frac{C\dot{x}}{\sqrt{(2ax - x^2)}} = \frac{(a + b)x\dot{x} - x^2\dot{x}}{\sqrt{(2ax - x^2)}}$;

E 2

dividing

dividing both sides of the equation by $\frac{\dot{x}}{\sqrt{(2ax-x^2)}}$ and collecting the terms $-2Ax^2 + 3Aax - Bx + Ba + C = (a+b)x - x^2$; equating the coefficients of the homologous terms, $A = \frac{1}{2}$, $3Aa - B = (a+b)$, or $B = 3Aa - (a+b) = \frac{a}{2} - b$, and $C = -Ba = -a\left(\frac{a}{2} - b\right)$; therefore $\int \dot{x} \cong \frac{1}{2}x \sqrt{(2ax-x^2)} + \left(\frac{a}{2} - b\right) \sqrt{(2ax-x^2)} - a\left(\frac{a}{2} - b\right) Q$, where Q is the circular arc, radius 1 and versed sine $= \frac{x}{2a}$.

J. W. of Leeds also answered this Question.

XXIII. QUESTION 143, by Mr. BAZLEY.

Suppose a thread of a given length to be fastened at one end to a given point on a perfectly smooth horizontal plane having a straight edge; and a weight attached to the other end of the thread and brought to the edge of the horizontal plane with the thread stretched; then if the weight be suffered to descend by the force of gravity from this position; it is required to determine the time in which the thread will have acquired any other possible assigned position upon the plane, together with the velocity of the descending weight in such position; also required the nature of the curve described by the descending weight?

FIRST SOLUTION, by Mr. BAZLEY.

Let AD (fig. 25, pl. 1.) be the edge of the horizontal plane, C the given point therein, CD the first position of the thread, CB any other position of it; and CA perpendicular to AB. Put $CA = a$, $AB = x$, and $g = 16\frac{1}{17}$ feet. Then by the resolution of forces, $\sqrt{(a^2 + x^2)} : x :: 2g : 2gx \div \sqrt{(a^2 + x^2)} =$ the force whereby the point B is accelerated along the edge of the plane BA: let the velocity in B $= v$; and, by the laws of variable forces, we have, $-2gx \div \sqrt{(a^2 + x^2)} = \frac{v\dot{v}}{v}$; where

where x is written negative, because v increases as x decreases; and taking the fluents, $-2g \sqrt{(a^2 + x^2)} = \frac{1}{2}v^2$, or, $-4g \sqrt{(a^2 + x^2)} = v^2$: but when $x = AD = r$, then $v = 0$; whence, by correction, $4g (\sqrt{(a^2 + r^2)} - \sqrt{(a^2 + x^2)}) = v^2$, or putting $\sqrt{(a^2 + r^2)} = CD = d$, we obtain $v = 2g^{\frac{1}{2}} \sqrt{\{d - \sqrt{(a^2 + x^2)}\}}$. Again, for the time, we shall have,

$$t = \frac{-\dot{x}}{2g^{\frac{1}{2}} \sqrt{\{d - \sqrt{(a^2 + x^2)}\}}}, \text{ where } \dot{x} \text{ is negative for the}$$

same reason as before. Now assume

$$\frac{-\dot{x}}{2g^{\frac{1}{2}} \sqrt{\{d - \sqrt{(a^2 + x^2)}\}}} = \dot{x} \times (Ax + Bx^2 + Cx^3 + Dx^4 + \&c.),$$

and reduce the whole expression by involution, &c. to simple terms, and we shall find

$$A = -\frac{1}{c^{\frac{1}{2}}}, B = -\frac{1}{4ac^{\frac{3}{2}}}, C = -\frac{3a-2c}{32a^2c^{\frac{5}{2}}}, D = -\frac{5a^2-6ca+4c^2}{128a^3c^{\frac{7}{2}}}, \&c.$$

$$\text{whence, taking the fluents, } t = Ax + \frac{Bx^3}{3} + \frac{Cx^5}{5} + \frac{Dx^7}{7} +$$

&c. wherein taking the values of $A, B, C, D, \&c.$ as found above, and making the series (by correction) to vanish when $x = 0$, we shall obtain the correct value of t .

And the velocity of the descending weight at any point will be readily obtained from what has been deduced; for the velocities in any direction are directly as the fluxions in the respective directions; therefore, fluxion of $BA = -x$: fluxion of $BW = -x\dot{x} \div \sqrt{(a^2 + x^2)} :: \sqrt{(a^2 + x^2)} : x ::$

$$v = 2g^{\frac{1}{2}} \sqrt{\{d - \sqrt{(a^2 + x^2)}\}} : 2g^{\frac{1}{2}} x \sqrt{\frac{d - \sqrt{(a^2 + x^2)}}{a^2 + x^2}},$$

which expresses the velocity of the descending weight at W .

As to the curve described by W , it is the equilateral hyperbola. For since BW is perpendicular to AB , it will still be so if we conceive the plane of the curve to be raised up, till it coincides with the given horizontal plane; then let CA be produced till $AG = CD$, draw GF parallel to AD , and produce BW to F . Now by the description of the curve, $CB + BW = CD = BF$ by construction; therefore $CB = FW$, and $CB^2 = FW^2 = AC^2 + AB^2$: let the curve meet CG in V ; then $CA = VG$, and $FW^2 = GV^2 + GF^2$, which is the property of the equilateral hyperbola whose centre is G and vertex V .

SECOND

SECOND SOLUTION, by Mr. LOWRY.

Let AD (fig. 25, pl. 1.) be the straight edge of the horizontal plane, C the point where the end of the thread is fastened, CD the position of the thread at the commencement of the motion, or when the weight is at D, the edge of the plane. Draw AC perpendicular to AD and put $AC = a$, $AD = b$, $CD = d$, $AB = x$, and $g = 32\frac{1}{2}$ feet the measure of the accelerative force of gravity. Then the force accelerating the thread towards A at the point B, being to the force in the direction CB, as AB to BC, we have, $CB : AB :: g : gx \div \sqrt{(a^2 + x^2)} =$ the accelerative force in the direction AB; therefore by the known principles of dynamics

$$\frac{\ddot{x}}{t^2} + g \cdot \frac{x}{\sqrt{(a^2 + x^2)}} = 0$$

multiply by $2\dot{x}$ and take the fluents, and we have

$$\frac{\dot{x}^2}{t^2} + 2g \sqrt{(a^2 + x^2)} = c.$$

To determine c , the velocity or $\frac{\dot{x}}{t}$ is $= 0$, when $x = b$ therefore c is $= 2g \sqrt{(a^2 + b^2)} = 2gd$, and

$$\frac{\dot{x}^2}{t^2} = 2gd - 2g \sqrt{(a^2 + x^2)}.$$

$$\text{or, } \frac{\dot{x}}{t} = (2g)^{\frac{1}{2}} \sqrt{\left\{ d - \sqrt{(a^2 + x^2)} \right\}}$$

$$\text{and } t = \frac{\dot{x}}{(2g)^{\frac{1}{2}} \sqrt{\left\{ d - \sqrt{(a^2 + x^2)} \right\}}}.$$

The fluent of this expression, may, by a very simple transformation, be exhibited by means of elliptic arcs, or it may be found by a series as is done at page 200 of Dr. Hutton's Select Exercices and conic sections.

Now $\frac{\dot{x}}{t} = (2g)^{\frac{1}{2}} \sqrt{\left\{ d - \sqrt{(a^2 + x^2)} \right\}}$ is the velocity in the direction DB, and the velocity in the direction BW is to the velocity in the direction DB as the fluxion of BW to the fluxion of DB; that is, as $\dot{x}$ to $x\dot{x} \div \sqrt{(a^2 + x^2)}$, therefore the velocity with which the weight descends is

$$= \sqrt{\frac{2gx^2}{a^2 + x^2}} \times \sqrt{\left\{ d - \sqrt{(a^2 + x^2)} \right\}}.$$

To

To find the nature of the curve described by the descending weight, put $BW = y$, then $CB = \sqrt{(a^2 + x^2)}$, and since $CB + BW = CD = d$, we have $BW = d - \sqrt{(a^2 + x^2)}$, and by reduction $x^2 = b^2 - 2dy + y^2$, an equation to the equilateral hyperbola.

XXIV. QUESTION 144, by Mr. CUNLIFFE.

Having given a triangle, it is required to determine a circle and concentric ellipse, together with the position of a right line passing through the said centre, such, that one of the angles of the triangle being always placed upon the right line, the other two angles shall be one in the circumference of the circle, and the other in the periphery of the ellipse?

FIRST SOLUTION, by Mr. LOWRY.

Let RFB (fig. 26, pl. 2.) be the given triangle, and take RH and CR each equal to the side FR. With the centre C, and radius CR, describe a circle RI; with the semiaxes CB, HB, describe an ellipse DBE, and through the given points C, F, draw the straight line CP; then RI is the circle, and DBE the ellipse which were to be found, and CP is the straight line given by position.

Let MKG be a triangle equal to RFB, having the angle K (F) placed at any point on the line CP, and the angle M (B) placed in the periphery of the ellipse; then we have only to prove that the point G (R) is in the circumference of the circle RI. Produce MG till $MT = CB$ and join TK; then the triangles MTK, BCF will be equal in all respects, and it is proved, in the second solution to the 88th. question in Vol. I, that the point T is in the axis CD, and that $TO = CH = 2CR$. But $MG = BR$ and $BC = MT$, therefore $TG = CR$; and, consequently, GO is also $=$ to CR, and TO is bisected in G. Therefore, because TCO is a right angle, G is the centre, and GC the radius of the circle which passes through the points T, C, O, that is, GC is $=$ to $\frac{1}{2}TO = CR$, and, consequently, the point G is in the circumference RI described with the radius CR.

SECOND SOLUTION, by Mr. CUNLIFFE, the Proposer.

Let CA, CB (fig. 27, pl. 2.) be two right lines given by position, and EF a right line of a given length having its extremes E and F always in the lines CA, CB; then the
locus

locus of any point G in EF will be an ellipse whose centre is C ; this is very well known. Draw ED , FD , respectively perpendicular to CA , CB meeting each other in D , and draw CD , which bisect in O , and join EO , FO , and GO .

Then CED and CFD being right angles, a circle whose diameter is CD will pass through E and F ; and CD is given, because EF and the angle ECF are given; hence $CO = OE = OF = \frac{1}{2}CD$ is given, therefore the locus of O is a circle whose centre is C . Wherefore the triangle EGO will have the angle E upon the right line CA , the angle O in the circumference of the circle whose centre is C and radius $CO = OE$; and the angle G in the periphery of an ellipse whose centre is C , and principal semi-axes $OC + OG$ and $OE - OG$; and the angle formed by the semi-transverse axis and the line CA equal to half the angle EOG ; this appears from my solution to question 113.

And the determination of the circle, concentric ellipse, and position of the right line passing through the common centre for a given triangle will easily follow.

For let EOG (fig. 28, pl. 2.) be a given triangle; E the angle to be placed on the right line; O the angle to be placed in the circumference of the circle; and G the angle to fall in the periphery of the ellipse.

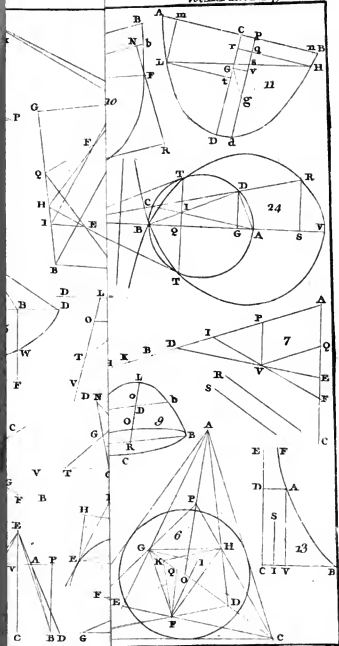
Produce GO till $OC = OE$ and join CE ; with the centre C and radius CO describe a circle; also with the centre C , semi-transverse GC and semi-conjugate equal to $OE - OG$, $OC - OG$, describe an ellipse; then the angle E being always placed on the right line CE and the angle O in the circumference of the circle whose centre is C and radius $CO = OE$ (the planes of the triangle, circle and ellipse coinciding), the angle G will fall in the periphery of the ellipse.

XXV. QUESTION 145, by Mr. I. I.

If from a point without an ellipse or an hyperbola, two tangents be drawn to the curve, and likewise two straight lines be drawn to the foci, the angle contained by one tangent and the line drawn to one focus will be equal to the angle contained by the other tangent and the line drawn to the other focus?

SOLUTION, by Mr. LOWRY.

Let PG , pG , be two tangents drawn from the point G to an ellipse or hyperbola; F , f , the foci, and let the lines GF , Gf , be drawn; then in the ellipse (fig. 29, pl. 2.) the angle PGF



$PGf = pGF$, and the angle $PGF = pGf$. And in the hyperbola (fig. 30, pl. 2.), fG being produced to L , the angle $PGL = pGF$, and the angle $PGF = pGL$.

Draw FQ , FH perpendicular to the tangents, meeting fP , fp (or these lines produced) in Q and H , and join QG , HG . Then, because the tangent PG bisects the angle QPF , it is evident that the right angled triangles PIF , PIQ , are equal in all respects; therefore $PF = PQ$; and the triangles GPQ , QPF , having two sides and the included angles respectively equal, they will be equal in every respect; therefore $GQ = GF$, and the angle $PGQ = PGF$. In the same way it is shewn that $GH = GF$, and the angle $pGH = pGF$; therefore GQ is $=$ to GH , and fQ is $=$ to fH (each being equal to the transverse axis AB), therefore the triangles fQG , fHG , are equal in every respect, and the angle $fGQ = fGH$.

Now, in the ellipse, the angle fGQ is $= PGf + PGF$, and the angle $fGH = pGf + pGF$; therefore $PGf + PGF = pGf + pGF$: take the angle fGF from each, and we have $2PGf = 2pGF$, or $PGf = pGF$. Let the angle fGF be added to each of these and then the angle $PGF = pGF$.

Again, in the hyperbola, because the angle $fGQ = fGH$, the angle QGL is $=$ to HGL : but the angle QGL is $= PGF + PGL$, and the angle HGL is $= pGF + pGL$; therefore $PGF + PGL = pGF + pGL$; take the angle FGL from each and we have $2PGL = 2pGF$, or $PGL = pGF$. Let the angle FGL be added to each of these and we have the angle $PGF = pGL$.

A property analogous to that which we have just demonstrated for the ellipse and hyperbola, belongs also to the parabola; for if tangents GP , Gp be drawn to a parabola whose focus is F , and Gf be a diameter, then the angle PGf will be equal to the angle pGF . (fig. 31, pl. 2.)

Let the lines be drawn as in the other cases, and join Pp , QH meeting Gf in f and S : then it is proved in the same way as before, that the angle $FGI = QGI$, the angle $FGK = KGH$, and $QG = GH$. But Pf is $= pf$, (because Pp is an ordinate to Gf), therefore QS is $= SH$; and the triangle QGH being isosceles, GS is perpendicular to QH , therefore the angle fGQ is $=$ to the angle fGH . The remaining part of the demonstration will now be the same as for the ellipse.

Mr. Bazley also favoured us with an answer to this question.

XXVI. QUESTION 146, by Mr. LOWRY.

Let there be any number of straight lines AP , BP , CP , DP , &c. given by position, and let Z be a given point without them; it is required to draw a circle through the

points P and Z, to intersect the given lines in A, B, C, D, &c, so that the sum of all the intercepted parts AP, BP, CP, DP, &c. may be equal to a given line?

FIRST SOLUTION, by Mr. LOWRY, the Proposer.

Draw PQ (fig. 32, pl. 2.) to bisect the angle APB and meet the circle in Q, then PQ is given by position. Assume any point a, as given, in PA, and draw ab parallel to PB meeting PQ in b, then b is a given point. Make bc parallel to PC, and equal to Pa; then c is a given point, and the line Pc drawn to meet the circle in R, is given by position. Make cd parallel to PD, and equal to Pb; then d is a given point, and the line Pd drawn to meet the circle in S, is given by position. Proceed in this manner till lines have been drawn parallel to all the lines given by position. Join AQ, BQ, CQ, RQ, SQ, RD, &c. then, because ab is parallel to PB, and the angle APQ (ABQ) = QPB (QAB), the triangles aPb, QBA, are equiangular, therefore $AB : AQ :: Pb : Pa$. Again, because bc is parallel to PC, the angle bcP is = to the angle RPC or RQC, and the angle bPc or QPR is = to QCR, therefore the triangles RCQ, Pbc, are equiangular;

and $RC : QR :: Pb : bc = Pa$; and $QC : QR :: Pc : Pb$.

It may be proved in the same way that the triangles RDC, Pdc, are equiangular, and, therefore, $SD : RS :: Pc : dc = Bb$, and $RD : RS :: Pd : cd$.

Now, Playfair's Euclid, Prop. E. B. VI.

$PA + PB : PQ :: AB : AQ :: Pb : aP :: RC : QR$, or $(PA + PB)QR = PQ \cdot RC = (\text{Prop. D. ibid.}) PR \cdot QC - PC \cdot QR$; therefore $(PA + PB + PC) QR = PR \cdot QR$,

or, $PA + PB + PC : PR :: QC : QR$; therefore $(PA + PB + PC)RS = PR \cdot SD = (\text{prop. D. ibid.}) PS \cdot RD - PD \cdot RS$, and $(PA + PB + PC + PD) RS = PS \cdot RD$,

or, $PA + PB + PC + PD : PS :: RD : RS :: Pd : cd$.

We may proceed in the same manner for any number of lines whatever, and the method of construction is now very obvious. Thus, if there be two lines AP, BP, and S be the given sum, let PQ be drawn to bisect the angle APB, and take QP to S in the given ratio of aP : Pb; then a circle described through the points, P, Q, Z, will be that required. If there be three lines AP, BP, CP, let PR be drawn as in the analysis, and take PR to S in the given ratio of Pb : Pc; describe a circle through the points P, R, Z, and the problem will be done. If there be four lines

AP,

AP, BP, CP, DP, let PS be drawn, and make PS to S in the given ratio of $cd : Pd$; then the circle passing through the points P, S, Z, will be the one required.

The circle, instead of passing through a given point Z, may either touch a straight line or a circle given by position, or have its centre in a line of any kind given by position.

The analysis of the problem will be nearly the same when the sum of the rectangles contained by the lines PA, PB, PC, PD, &c. and given lines a, b, c, d, &c. is equal to a given space.

The following porism may also be immediately deduced from what has been done above.

Let there be any number of straight lines PA, PB, PC, PD, &c. given by position and intersecting in the same point P; a point may be found, such, that if any circle whatever be described through the point P and the point found, and meeting the straight lines given by position in the points A, B, C, D, &c.; the sum of the intercepted parts AP, BP, CP, DP, &c. shall be equal to a given line: or the sum of the rectangles $a \cdot AP$, $b \cdot BP$, $c \cdot CP$, $d \cdot DP$, &c. shall be equal to a given space.

SECOND SOLUTION, by Mr. J. W. of Leeds.

From the given point Z (fig. 33, pl. 2.) draw $Z\alpha$, $Z\beta$, $Z\gamma$, $Z\delta$, &c. perpendicular to PA, PB, PC, PD, &c. and join PZ; then having bisected PZ in X with the indefinite perpendicular Xx, and taken L equal to the line, let OX be taken in Xx so that $Z\alpha + Z\beta + Z\gamma + Z\delta$, &c. : $L - P\alpha - P\beta - P\gamma - P\delta$, &c. :: PX : OX; join OP, OZ, and a circle described about the centre O with the radius OP or OZ will be that which was required to be described.

Let the circles meet the straight lines given by position in A, B, C, D, &c. and join ZA, ZB, ZC, ZD, &c. It is evident that the triangles $Z\alpha A$, $Z\beta B$, $Z\gamma C$, &c. are each of them similar to the triangle PXO: hence

$PX : OX :: Z\alpha : A\alpha :: Z\beta : B\beta :: Z\gamma : C\gamma :: Z\delta : D\delta$, &c. wherefore $Z\alpha + Z\beta + Z\gamma + Z\delta$ &c. : $A\alpha + B\beta + C\gamma + D\delta$ &c. :: PX : OX; but by the construction

$Z\alpha + Z\beta + Z\gamma + Z\delta$ &c. : $L - P\alpha - P\beta - P\gamma - P\delta$ &c. PX : OX; therefore $A\alpha + B\beta + C\gamma + D\delta$ &c. = $L - P\alpha - P\beta - P\gamma - P\delta$ &c. wherefore $A\alpha + B\beta + C\gamma + D\delta$ &c. + $P\alpha + P\beta + P\gamma + P\delta$ &c. = L; that is, the sum the lines PA, PB, PC, PD, &c. is equal to L the given line by construction.

XXVII. QUESTION 147, by Mr. WALLACE, R. M. College.

If x denote an arch of the meridian, and y denote the meridional parts corresponding to the latitude x , then

$$y = x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{725678} + \&c.$$

and by reverting the series

$$x = y - \frac{y^3}{6} + \frac{y^5}{24} - \frac{61y^7}{5040} + \frac{277y^9}{725678} - \&c.$$

where it is remarkable that the numeral coefficients of the powers of x are the same as those of the like powers of y , and the signs of the terms of the one series are all +, but those in the other series + and - alternately. Shew the reason of these peculiarities?

SOLUTION, by Mr. WALLACE, the Proposer.

It is well known that the relation between x and y is expressed by the equation

$$y = \int \frac{\dot{x}}{\text{cof. } x} = \frac{1}{2} \text{ hyp. log. } \frac{1 + \text{fin. } x}{1 - \text{fin. } x},$$

therefore, putting e for the number whose hyp. log. = 1,

$$e^{2y} = \frac{1 + \text{fin. } x}{1 - \text{fin. } x}$$

$$\text{and hence fin. } x = \frac{e^{2y} - 1}{e^{2y} + 1},$$

$$\text{and cof. } x = \frac{2e^y}{e^{2y} + 1} = \frac{1}{\frac{1}{2}(e^y + e^{-y})},$$

and as from the fluxional equation $\dot{y} = \frac{\dot{x}}{\text{cof. } x}$ we have

$$\text{cof. } x = \frac{\dot{x}}{\dot{y}}, \text{ therefore}$$

$$\frac{\dot{x}}{\dot{y}} = \frac{1}{\frac{1}{2}(e^y + e^{-y})}, \text{ and } \dot{x} = \frac{\dot{y}}{\frac{1}{2}(e^y + e^{-y})};$$

But

$$\text{But } \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.$$

$$\text{and } e^y = 1 + y + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} + \&c.$$

$$\text{and } e^{-y} = 1 - y + \frac{y^2}{1.2} - \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} - \frac{y^5}{1.2.3.4.5} + \&c.$$

$$\text{and therefore } \frac{1}{2}(e^y + e^{-y}) = 1 + \frac{y^2}{1.2} + \frac{y^4}{1.2.3.4} + \frac{y^6}{1.2.3.4.5.6} + \&c.$$

$$\text{Put } \frac{1}{1.2} = a, \frac{1}{1.2.3.4} = b, \frac{1}{1.2.3.4.5.6} = c, \&c.$$

then substituting these values of $\cos x$ and $\frac{1}{2}(e^y + e^{-y})$ in the fluxional equations

$$\dot{y} = \frac{\dot{x}}{\cos x}, \quad \dot{x} = \frac{\dot{y}}{\frac{1}{2}(e^y + e^{-y})}$$

they become

$$\dot{y} = \frac{\dot{x}}{1 - ax^2 + bx^4 - cx^6 + \&c.},$$

$$\dot{x} = \frac{\dot{y}}{1 + ay^2 + by^4 + cy^6 + \&c.}.$$

Now if v denote any quantity, by known methods we find

$$\frac{1}{1 - av + bv^2 - cv^3 + \&c.} = 1 + Av + Bv^2 + Cv^3 + \&c.$$

where $A, B, C, \&c.$ denote certain constant numbers that are independent of the value of v , therefore the same equation must hold true whatever quantity be substituted instead of v . If we substitute x^2 then we get

$$\frac{1}{1 - ax^2 + bx^4 - cx^6 + \&c.} = 1 + Ax^2 + Bx^4 + Cx^6 + \&c.$$

and if we put $-y^2$ instead of v we get

$$\frac{1}{1 + ay^2 + by^4 + cy^6 + \&c.} = 1 - Ay^2 + By^4 - Cy^6 + \&c.$$

therefore the two equations

$$\dot{y} = \frac{\dot{x}}{1 - ax^2 + bx^4 - cx^6 + \&c.}$$

$$\dot{x} = \frac{\dot{y}}{1 + ay^2 + by^4 + cy^6 + \&c.}$$

are

are equivalent to these other equations

$$\dot{y} = \dot{x} + Ax^2\dot{x} + Bx^4\dot{x} + Cx^6\dot{x} + \&c.$$

$$\dot{x} = \dot{y} - Ay^2\dot{y} + By^4\dot{y} - Cy^6\dot{y} + \&c.$$

from which, by taking the fluents, and putting α for $\frac{1}{2}A$, β for $\frac{1}{2}B$, γ for $\frac{1}{2}C$, &c. we get

$$y = x + ax^3 + \beta x^5 + \gamma x^7 + \&c.$$

$$x = y - ay^3 + \beta y^5 - \gamma y^7 + \&c.*$$

And hence it appears that the two series expressing the values of x and y in terms of each other must necessarily have the properties described in the question.

XXVIII. QUESTION 148, by Mr. I. R.

Let two circles, one of which is included within the other, be given in a plane, and let there be given a point, situated within both the circles, in the line joining the two centres; it is required to describe a circle, from the given point as a centre, to meet the peripheries of the given circles, so that the angle contained by two straight lines drawn from the given point to the two points of concurrence may be the least possible.

SOLUTION, by Mr. W. WALLACE, R. M. College.

It may be presumed that, in framing this question, the ingenious proposer had in view this well known astronomical problem, "To determine the time of the year, when the twilight is the shortest possible, in a given latitude:" for if we consider the greater of the two given circles, as the stereographic projection of the horizon of a given place upon the plane of the equator, and the lesser circle, that of a circle parallel to the horizon and 18° below it, and the given point as the projection of the pole of the world, then the circle to be found will evidently be the parallel which the sun describes on the day of the year, when the twilight is the shortest possible at that place. The following answer to the question, may therefore be considered as a geometrical solution of the above mentioned astronomical problem.

Fig. 34, pl. 2. Let A be the centre of the greater, and B that

* For the law according to which these coefficients α , β , γ , &c. are formed, see page 54, vol. 1, of the Repository, New Series.

that of the lesser of the two given circles, and P the given point in the line joining their centres. Conceive the thing to be done, that is, suppose a circle described on P as a centre to meet the lesser of the given circles in D , and the greater in δ , so that DP , δP being drawn, the angle $DP\delta$ is the least possible. Let PB produced meet the greater circle in E and the lesser in F ; on P as a centre with the distance PB describe a circle; draw $P\beta$ a radius of this circle, so that the angle $BP\beta$ may be equal to $DP\delta$, and join $\beta\delta$, then the angle $DP\beta$ being equal to $BP\beta$, by taking the common angle $DP\beta$ from both, the remaining angles DPB , $\delta P\beta$ will be equal, and as $DP = \delta P$ and $BP = \beta P$, the triangles DPB , $\delta P\beta$ having two sides and the included angle of the one respectively equal to two sides and the included angle of the other, their bases BD , $\beta\delta$ are equal, but BD is given, and, being equal to BF , must be less than BE , therefore $\beta\delta$ is less than BE . Hence we see the possibility of $DP\delta$, or its equal the angle $BP\beta$, or the arch $B\beta$ (which is proportional to this last angle) having a minimum value, and it is also pretty evident that when the arch $B\beta$ is the least possible a circle described on β as a centre with a distance equal to $\beta\delta = BD$ must touch the exterior of the two given circles in δ , therefore δB when produced must pass through its centre A , and as $A\delta$ and $\delta\beta$ are each given in magnitude their difference $A\beta$ is given in magnitude, thus the point β is given, and the line $A\beta$ is given by position, therefore the point δ is given. This analysis immediately suggests the following construction.

Having described a circle on P as a centre with PB as a radius, from A place a line $A\beta$ equal to $AE - BF$ (the difference of the radii of the given circles) to terminate at β , a point in its circumference, produce $A\beta$ to meet the exterior of the two given circles in δ , then a circle described on P as a centre so as to pass through δ will meet the other given circle in D , so that δP , DP , being drawn, the angle $DP\delta$ will be a minimum as required.

SECOND SOLUTION, by Mr. LOWRY.

Fig. 35, pl. 2. Let HR , AB be the diameters, and C , O , the centres of the given circles, p the given point, and IK an arc of the circle required. Draw Ip , Kp , and, instead of supposing the angle IpK to be the least possible, let it be a given angle. Draw pG to make the angle OpG equal to that angle, and take pG equal to pO ; then G will be a given point. Join GK and OI ; then, if the angle GpI be added to each of the equal angles IpK , OpG , the angle OpI will be equal to the angle GpK , and the side pI being equal to the side pK , and the side Op to the side pG , the triangles pIO , pKG , will be equal in
all

all respects; therefore GK is equal to OI, a given line; and G being a given point, the locus of the point K will be a circle mKn, given by position.

Now, when the circles mKn, HKR, intersect each other, there will be two points K, where the angle IpK will be equal to the given one: but when they only touch each other, there can be one point K only, and the angle IpK will then be the least possible. For if it was less, the circle mKn could not meet the circle HKR, and the problem would be impossible.

When the circles touch at K, the line KG passes through the centre C, and CG being equal to OI or OA, and CK equal to CH, CG will be equal to the difference of the radii CH, OA (equal to CO + AH), and Gp being equal to pO we have the following simple construction:

With the distances pG (pO), and GC (CO + AH), and centres p, C, describe two arcs to intersect at G, produce CG to meet the exterior circle in K, and pK is the radius of the circle required.

The preceding analysis and construction may be applied to the solution of an astronomical problem which has often engaged the attention of mathematicians; viz. To find on what day of the year the duration of twilight is the shortest in a given latitude.

For, conceive a plane, perpendicular to the horizon II'R' (fig. 36, pl. 2.), to pass through the poles P, P', and meet the plane of the equator in E, Q, and the plane of a small circle 18° below the horizon, and parallel to it, in a, b. Draw PH', Pa, PP', Pb, PR', meeting the plane of the equator in H, A, p, B, R; then these points will evidently be the projections of the points H', a, P', b, R' on that plane respectively. Also the angle H'PP' being measured by half the arc H'P', the tangent Hp of the angle H'PP' (to the radius pP) will be the tangent of half the arc H'P' or PR', the latitude of the place. For a like reason pR will be the tangent of half the arc P'R' = tang. $\frac{1}{2}$ (90° - $\frac{1}{2}$ lat.); pA the tangent of half the arc aP' = tang. $\frac{1}{2}$ (lat. - 18°) = tang. ($\frac{1}{2}$ lat. - 9°); and pB = the tangent of half the arc P'b = tang. $\frac{1}{2}$ (90° - $\frac{1}{2}$ (lat. + 18°)) = tang. (81° - $\frac{1}{2}$ lat.).

Bisect HR in C, and join PC; then because HPR is a right angle, PC is = to CH or CR, and the angle HCP = to twice the angle PRC = to twice the angle HPP = to the angle P'pH' = to the angle NpQ; therefore pC, the tangent of the angle PPC, is the tangent of the arc P'N, the co-latitude; and PC is the secant of the co-latitude.

Now, let the distances from p (in fig. 36,) be transferred to fig. 35, by taking pQ = pP for the radius of the primitive circle,

circle, pC = the co-tangent, and CH = the co-secant of the given latitude: with the centre C and distance CH , describe a circle HKB , which will represent the horizon, when projected on the plane of the equator. Again, take Ap = the tang. $(\frac{1}{2} \text{ lat} - 9^\circ)$, and Bp = $(81^\circ - \frac{1}{2} \text{ lat.})$ and on the diameter AB describe a circle AIB which will be the projection of the small circle at 18° from the horizon, where the twilight begins or ends. Bisect AB at O , and draw CK as in the foregoing construction; then the distance pK will be the tangent of half the co-declination. For the arc KI , described about the pole p , will be the projection of the parallel of declination, pK and pI the projections of the azimuth circles at the beginning and end of twilight, and the angle KpI will be equal to the angle included between those circles, and which angle is proportional to the duration of twilight, and therefore, when the angle KpI is the least possible, the duration of twilight will be the shortest.

When the latitude is 18° , the points A and p coincide, and when the latitude is less than 18° , the point A will fall between p and B , but the construction will still be the same as the above. When Hp is = to pB , the points K, I , will coincide with H, B , and tang. $\frac{1}{2} \text{ lat.} = \text{tang. } (81^\circ - \frac{1}{2} \text{ lat.})$ or the latitude = 81° , and the twilight will continue twelve hours.

The calculation may be easily made out from the construction, for in the triangle pGC , the three sides are given to find the angle at G , and then in the triangle pGK , two sides and the included angle are given to find the side pK .

XXIX. QUESTION 149, by MECHANICUS.

To place a straight rod, or beam, loaded with any given weights, so that it may be in equilibrio, resting upon an upright prop, and one end touching a smooth vertical wall given by position?

FIRST SOLUTION, by Mr. LOWRY.

Let AB (fig. 37, pl. 2.) represent the vertical plane or wall, ID the upright prop, KB the position of the rod when it rests in equilibrio, and G the centre of gravity of the rod and weights. Draw GE parallel, and BE perpendicular to AB , and join EI . Then, the forces that act on the rod are these three; the force of gravity, equal to the whole weight, acting at G in the vertical direction GE ; the force arising from the pressure of the prop against the rod at I , acting in a direction perpendicular to

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G

KB;

KB; and, thirdly, the force arising from the pressure of the wall against the end of the rod at B, which acts in the direction BE. And that the rod may rest in equilibrio, it is necessary that the directions of these forces tend to the same point; therefore EI will be the direction of the force acting at I, or EI will be perpendicular to the rod KB. Because ID is parallel to GE, the forces at G, I, and B, are proportional to the lines ID, IE, and ED; or, if IE represent the absolute force at I, and be resolved into the two forces ED, ID; then the force ED is that part of the force IE which is equal and opposite to the pressure at B, and prevents the rod from moving horizontally; and the force ID is that part of the force IE which is equal and opposite to the force of gravity at G, and hinders the rod from descending in a vertical direction.

To find the position of the rod, let IC be drawn perpendicular to AB, and produce EI to meet the plane at A; then by similar triangles GB:BI::EB:BD=IC::AB:AC::AB·BC:AC·BC; but AB·BC = BI², and AC·BC = IC²;

therefore GB : BI :: BI² : IC², or BI³ = GB·IC².

Put l = to the length of the rod; b = to IC, the distance of the prop from the wall; m = the weight of the rod; $w, w', \&c.$ the weights placed at the distances $d, d', \&c.$ from the end B of the rod; a = GB, the distance of the common centre of gravity of the rod and weights from the same end B; and x = IB. Then, by the ordinary rule for finding the centre of gravity of weights placed in a straight line, we have

$$GB = a = \frac{\frac{1}{2}lm + dw + d'w' + \&c.}{m + w + w' + \&c.} = \frac{\frac{1}{2}lm + W'}{m + W'}$$

where $W' = dw + d'w' + \&c.$ and $W = w + w' + \&c.$

$$\text{Therefore } x^3 = b^2 \times \frac{\frac{1}{2}lm + dw + d'w' + \&c.}{m + w + w' + \&c.} = b^2 \times \frac{\frac{1}{2}lm + W'}{m + W'}$$

If $m = 0$, or the rod has no weight, and a single weight w only be placed at the extremity of the rod; then $x^3 = b^2$

$\times \frac{dw}{w} = b^2 \times l$, which shews that the position of the rod will be the same, whatever the weight w may be, which is affixed to the end of it.

To compare the pressure at I and B with the sum of the weights $m + w + w' + \&c.$ or the force at G, let P, P' be put to denote these pressures respectively; then the triangles IBC,

$$\text{IED, being similar, } IC : IB :: ID : IE :: m + W : P = \frac{IB}{IC} \times$$

($m + W$)

$$(m + W) = \sqrt[3]{\frac{a}{b}} \times (m + W); \text{ and } IC : BC :: ID : IE ::$$

$$m + W : P = \frac{BC}{IC} \times (m + W) = \sqrt[3]{\frac{a^2b}{b}} \times \frac{m + W}{\sqrt{b}}.$$

This solution is on the supposition that the plane or wall is perfectly smooth, or void of friction: but if the friction be taken into account, and be supposed proportional to the pressure, the solution may then be as follows. Draw BF (fig. 38, pl. 2.) perpendicular to BA, and take BL to BF, in the given proportion of the friction to the pressure ($n : 1$); complete the parallelogram BFRL, and draw BR meeting CE, a parallel to AB at E; then since the friction opposes the ascent of the rod at B, it may be considered as a force acting in the direction BL; therefore BR will be the direction of a force, equivalent to the two forces of friction and pressure acting in the directions BL, BF. Let EN be drawn parallel to BF meeting ID produced in N; then, the rod is kept in equilibrio, in the same manner as before, by three forces acting in the directions GE, IE, BE; and proportional to the lines ID, IE, ED; or, if IE represent the absolute force at I, and be resolved into two forces EN, DN: then EN will be equal and opposite to the force of pressure at B, and will hinder the rod from moving in a horizontal direction; also DN will be equal to the force arising from the friction of the plane and will assist the force of gravity DI in opposing the vertical force IN, so as to prevent the rod from ascending or descending in a vertical direction,

To find the position of KB, draw IP parallel to EB; then by similar triangles, &c. as in the former case, we have $BI^3 = GB \cdot BC \cdot AP$. But $AP = AC - PC$, and $BC \cdot AP = BC (AC - PC) = IC^3 - BC \cdot PC$, therefore $BI^3 = GB (IC^3 - BC \cdot PC)$.

Because the triangles ICP, RLB, are similar, $PC : IC :: BL : LR$ or $BF :: n : 1$, therefore $PC = n \cdot IC = nb$, and $bc = \sqrt{c^2 - b^2}$.

$$\text{Therefore } x^3 = a \times \{b^3 - nb \sqrt{x^2 - b^2}\}^{\frac{1}{2}},$$

$$\text{or } x^6 - 2ab^2x^3 - n^2a^2b^2x^2 + (n^2 + 1)a^2b^4 = 0.$$

$$\text{Substitute } \frac{\frac{1}{2}lm + dw + d'w' + \&c.}{m + w + w' + \&c.}, \text{ or its equal } \frac{\frac{1}{2}lm + W'}{m + W}$$

for a in this equation, and we have

$$x^6 - 2b^2 \times \frac{\frac{1}{2}lm + W'}{m + W} \times x^3 - \{n^2b^2x^2 - (n^2 + 1)b^4\} \times \left(\frac{\frac{1}{2}lm + W'}{m + W}\right)^2 = 0.$$

If a single weight w only be fixed at the end of the rod, the expression becomes

$$x^6 - 2lb^2 \left(\frac{\frac{1}{2}m + w}{m + w} \right) x^3 - n^2 l^2 b^2 \left(\frac{\frac{1}{2}m + w}{m + w} \right)^2 x^2 + (n^2 + 1) l^2 b^2 \left(\frac{\frac{1}{2}m + w}{m + w} \right)^2 = 0$$

where if $m = 0$, then $x^6 - 2lb^2 x^3 - n^2 l^2 b^2 x^2 + (n^2 + 1) l^2 b^4 = 0$, which shews that in this case the equilibrium of the rod is not at all affected by the weight w .

If $n = 0$, or the friction ceases, then

$$x^6 - 2b^2 \times \frac{\frac{1}{2}l/m + W'}{m + W} \times x^3 + (n^2 + 1) b^4 \times \left(\frac{\frac{1}{2}l/m + W'}{m + W} \right)^2 = 0;$$

or $x^3 = b^2 \times \frac{\frac{1}{2}l/m + W'}{m + W}$, as determined before.

To find the pressures at I and B, put f to denote the force of friction; then $IC : BC :: ID + DN : EN :: m + W + f : P'$; but f is $= n \cdot P'$, by hypothesis; therefore $IC : BC ::$

$m + W + n \cdot P' : P'$; or, $P' = (m + W) \times \frac{\sqrt{(x^2 - b^2)}}{b - n \sqrt{(x^2 - b^2)}}$, and

$f = (m + W) \times \frac{n \sqrt{(x^2 - b^2)}}{b - n \sqrt{(x^2 - b^2)}}$. Again $BC : IB :: EN :$

$EI :: P' : P = P' \times \frac{IB}{BC} = (m + W) \times \frac{x}{b - n \sqrt{(x^2 - b^2)}}$.

From hence we may easily determine what force is necessary to be applied at B, in the direction BL, so as to retain the rod in any assigned position. For, from the equation above (*), we find

$n \sqrt{(x^2 - b^2)} = \frac{ab^2 - x^3}{ab}$, and substituting this in the value

of f , we have $f = (m + w + w' + \&c.) \times \frac{ab^2 - x^3}{x^2}$.

Some account of this problem is given in the 6th vol. of the Edinburgh Review. It appears to have been first resolved by Euler, in the Memoirs of the Academy of Berlin, Tome vi. and afterwards by Signor Fontana, in *Memoire di Matematica e Fisica della Societa Italiana di Scienze*, 1802.

Euler considered the case only when the rod was without weight, the plane void of friction, and the weight attached to the extremity of the rod. His solution is founded on a principle or law of nature introduced by Maupertuis, which was, that, "in preserving the repose of the universe, and in performing its various motions, nature uniformly employs those means which require the least expence of force."

Fontana, not satisfied with this principle of Maupertuis, resolved the problem by the ordinary principles of statics and the arithmetic

arithmetic of lines, and rendered it more general by supposing the rod to have weight, and the friction of the plane to be equal to n times the pressure. The result which he obtains is the same as determined above when the weight is placed at the extremity of the rod, and agrees with Euler's, when the rod is without weight and the plane does not exert any friction.

If the plane, instead of being vertical, be inclined to the horizon in a given angle, the problem will be rather more general. Draw BE (fig. 39, pl. 2.) perpendicular to the plane BA, to meet the vertical line GE in E, and join EI: then it appears as before, that EI is perpendicular to GB, and that the forces at G, I, and B, act in the directions GE, EI, and EB, and are proportional to the lines ID, EI, and ED.

To find the position of the rod, draw BV parallel to EG, or DI, meeting IC (a perpendicular to BA) in Q and EI produced in V; draw also IN perpendicular to BV, meeting the plane in A. Then, by similar triangles, &c. as in the first case, we obtain $IB^2 = GB \cdot BN \cdot VQ$; but $BN \cdot VQ = (BV - BQ) \cdot BN = BI^2 - BN \cdot BQ$, and because the angles at C and N are right angles, the points C, Q, N, A, are in a circle; therefore $BN \cdot BQ = BA \cdot BC = BC^2 + BC \cdot AC = BN \cdot VQ = IC^2 - AC \cdot BC$; therefore $BI^2 = GB (IC^2 - AC \cdot BC)$.

Let k be the angle which the plane makes with the horizon; then because IA is perpendicular to BV, or parallel to the horizon, the angle IAC is $= k$, and $CA = b \times \cot. k$, therefore $AC \cdot BC = b \cot. k \sqrt{(x^2 - b^2)}$ and $x^3 = a \{ b^2 - b \cot. k \sqrt{(x^2 - b^2)} \}$, or $x^6 - 2ab^2x^3 - b^2a^2 \cot. kx^2 + a^2b^4 - a^2b^4 \cot. k = 0$.

When k is $= 90^\circ$, or the plane is vertical, $\cot. k = 0$, and the equation becomes $x^6 - 2ab^2x^3 + a^2b^4 = 0$, or $x^3 = ab^2$ as before.

When the plane exerts friction, let BF (fig. 40, pl. 2.) be drawn perpendicular to the plane, and take $BF : BL :: 1 : n$; complete the parallelogram BFRL, and draw BR meeting the vertical line GE in E, and join EI; then EI will be perpendicular to GB, and the lines ID, EI, and ED, will be proportional to the forces acting on the rod at G, I, and B, in the directions GE, IE, and BE. Draw IP parallel to BE meeting BA in P, and BV in S, and draw ST perpendicular to BP; then by similar triangles, &c. as before, $BI^2 = GB \cdot BN \cdot VS$; but the points N, S, T, A, being in a circle, $BN \cdot BS = BA \cdot BT$, and $BN \cdot VS = BN (BV - BS) = BI^2 - BA \cdot BT$; therefore $BI^2 = GB (BI^2 - BA \cdot BT)$. Because the angle BST $= BAN = k$, $BT = ST \times \tan. k$, and $TP = TS \cdot n$, therefore $BP = ST$

$(\tan. k + n)$, and $BT = BP \times \frac{\tan. k}{\tan. k + n}$; and $BA \cdot BT =$

BA · BP

$$BA \cdot BP \cdot \frac{\tan. k}{\tan. k + n} = (BC + CA) (BC + CP) \times \frac{\tan. k}{\tan. k + n}$$

$$= \left\{ BC^2 + BC (CA + CP) + CA \cdot CP \right\} \times \frac{\tan. k}{\tan. k + n}. \text{ But}$$

$CA = b \cdot \cot. k$, and $CP = n \cdot b$; therefore, if y be put $= BC$, we have

$$x^2 = a \left\{ y^2 - [y^2 + y(b \cot. k + n \cdot b) + \cot. k \cdot n b^2] \times \frac{\tan. k}{\tan. k + n} \right\}.$$

Exterminating y by means of the equation $x^2 = b^2 + y^2$ we have an equation of the 6th power, containing only x and known quantities, and which, when $n = 0$, becomes the same as was determined in the last case.

When the centre of gravity G is placed between the prop and the plane, it is evident that the rod cannot rest in equilibrio, except some other force be opposed to the end of the rod so as to prevent it from sliding along the plane. To find what this force must be, so that the rod may rest in any assigned position, we shall first suppose the plane to be horizontal, as in fig. 41, pl. 2, and that KB is the rod, making a given angle k with the horizontal plane BD ; draw BL and RG perpendicular to BD ; and IE , BS perpendicular to BL meeting GR and ID produced in E , P , and S . Then, since the directions of the forces acting on the rod at I and G meet in E , it is evident that EB must be the direction of a force equivalent to the two forces acting at B in the directions BL and BR ; or, if EP represent the absolute force of gravity acting at G then BP will represent the pressure at I ; ER the pressure at B ; and BR the force f required.

Because $BR = GB \cdot \cos. k$, and $EP = IS = \frac{IB}{\sin. k} = \frac{ID}{\sin.^2 k}$;

therefore $BR : EP :: GB \cdot \cos. k : \frac{ID}{\sin.^2 k}$, or $BR = EP \times \frac{GB}{ID} \times \cos. k \cdot \sin.^2 k$; that is $f = (m + W) \times \frac{GB}{ID} \times \cos. k \cdot \sin.^2 k$

$$\sin.^2 k = \frac{\frac{1}{2}lm + W'}{ID} \times \cos. k \cdot \sin.^2 k.$$

Again, since $BP = \frac{BR}{\sin. k}$, P is $= \frac{\frac{1}{2}lm + W'}{ID} \cos. k \cdot \sin. k$,

the pressure at I ; and $ER = PE - PR = PE - BR \times \frac{\cos. k}{\sin. k}$,

therefore $P' = m + W - \frac{\frac{1}{2}lm + W'}{ID} \times \cos.^2 k \cdot \sin. k$,

the pressure at B .

When

When the weight of the rod only is taken into the account (supposing it to be of an uniform thickness), the problem is then the same as the first part of the prize question, in the Gentleman's Diary

for 1802; and in this case $f = \frac{\frac{1}{2}lm}{ID} \times \cos. k \cdot \sin.^2 k = m \times \frac{l}{2} \times \frac{BD \cdot DI}{IB^3}$; $P = m \times \frac{l}{2ID} \times \cos. k \cdot \sin. k$; and $P' = m \times (ID - \frac{1}{2}l \cos.^2 k \cdot \sin. k)$.

If the friction of the plane be equal to n times the pressure as in the other cases, take $BT : BF :: 1 : n$; complete the parallelogram $BFTH$, and draw EQ parallel to the diagonal HB ; then RQ will represent the force of friction, and QB the force required. The triangles ERQ , BHF , (or BHI) are evidently similar, and $FB = n \cdot BT$; therefore $RQ = n \cdot RE = n \cdot P'$, and $BQ = BR - RQ = f - n \cdot P'$ the force required.

When BH coincides with BE , Q will coincide with B , and the friction will then be sufficient to prevent the rod from slipping along the plane without the assistance of any other force.

Again when the plane BD (fig. 42, pl. 2.) makes a given angle with the horizontal plane BV , draw ID and GR perpendicular to the horizon, and draw also IE , BS , perpendicular to IB meeting GR and ID produced in E , P , and S . Also draw BL and EQ perpendicular to the plane BD and join BE : Then, if EP (or IS) represent the force at G , BP will represent the force of pressure at I , EQ the force of pressure at B , and BQ the force required, acting in the direction BQ .

Put $h =$ the angle DIB , $g =$ the angle IDB , $k =$ the angle DBI , and $p =$ the angle DPV , which the plane makes with the horizon. Then $BR = GB \frac{\sin. k}{\sin. g}$ and $IS = \frac{IB}{\cos. h}$, therefore

$RB : IS$ or $EP :: GB \cdot \frac{\sin. k}{\sin. g} : \frac{IB}{\cos. h}$, or $BR = EP \times \frac{GP}{BI} \times \frac{\sin. h \cdot \cos. h}{\sin. g}$; but the angle RBP is the complement of the angle k , and the angle BPR is the complement of the angle h ; therefore $PR = \frac{BR \times \cos. k}{\cos. h}$, and $ER = EP - PR = EP - BR \times \frac{\cos. k}{\cos. h}$; but $QR = ER \cdot \sin. p$; therefore $QR = EP \cdot \sin. p - BR \times \frac{\cos. k \cdot \sin. p}{\cos. h}$.

Hence

Hence $BQ = BR + QR = BR + \left(EP - BR \times \frac{\text{cof. } k}{\text{cof. } h} \right) \cdot \text{fin. } p$
 $= EP \left\{ \text{fin. } p + \left(1 - \frac{\text{cof. } k \cdot \text{fin. } p}{\text{cof. } h} \right) \cdot \frac{\text{fin. } h \cdot \text{cof. } k}{\text{fin. } g} \cdot \frac{GB}{IB} \right\},$
 or $f = (m + W) \cdot \left\{ \text{fin. } p + \left(\frac{\text{fin. } h \cdot \text{cof. } h}{\text{fin. } g} - \frac{\text{cof. } k \cdot \text{fin. } h \cdot \text{fin. } p}{\text{fin. } g} \cdot \frac{GB}{IB} \right) \right\}$
 the force required.

Again $BP = BR \times \frac{\text{fin. } g}{\text{cof. } h}$, and $P = (m + W) \times \text{fin. } h \cdot \frac{GB}{IB}$
 $= \frac{\frac{1}{2}lm + W'}{BI} \times \text{fin. } h$, the pressure at I.

Also $EQ = ER \cdot \text{cof. } p = \left(EP - PR \cdot \frac{\text{cof. } k}{\text{cof. } h} \right) \times \text{cof. } p$, and P
 $= (m + W) \times \text{cof. } p - \frac{\frac{1}{2}lm + W'}{BI} \times \frac{\text{fin. } h \cdot \text{cof. } k \cdot \text{cof. } p}{\text{fin. } g}$,
 the pressure at P.

When $p = 0$, $g = 90^\circ$, and ID is put for its equal $IB \times \frac{\text{fin. } k}{\text{fin. } g}$,
 the expressions for the forces and pressures, are precisely the
 same as determined for the horizontal plane.

When the friction of the plane is taken into the account,
 we have only to calculate the values of the forces f and P , as
 before, and then $f - \pi l'$ will be the force required.

SECOND SOLUTION, by Mr. CUNLIFFE.

Let the straight line HE (fig. 43, pl. 2.) represent the rod,
 or beam, loaded with the weights; G its centre of gravity;
 the point P the prop, and EB the vertical plane.

Through P and G draw PB, GT, perpendicular to the vertical
 plane. When the rod is in equilibrium in the circumstances
 mentioned in the question, the centre of gravity G will be at
 the highest point to which it can ascend: that is, BT will be a
 maximum in that circumstance, this is a well known statical
 principle.

Put $EG = a$, $PB = b$, and $PE = x$; then $GP = a - x$,
 and $EB^2 = x^2 - b^2$, and because PB and GT are parallel,
 $PE^2 : EB^2 :: GB^2 : BP^2 = \frac{EB^2 \times GP^2}{PE^2} = \frac{(x^2 - b^2)(a - x)^2}{x^2}$

=

$= (a-x)^2 - \frac{b^2(a-x)^2}{a}$ which is to be a maximum. Putting

its fluxion $= 0$, and reducing the equation, we find $x = \sqrt[3]{ab^2}$.

The problem will be something more general, if enunciated as follows: "To place a straight rod, or beam, loaded with any given weights, so that it may be in equilibrio, resting on an upright prop, and one end touching a smooth inclined plane."

Let HE (fig. 44, pl. 2.) represent the rod, or beam, loaded with the given weights, and G its centre of gravity; the point P the prop, and EB the smooth inclined plane. Through P and G, draw PB, GT, parallel to the horizon, meeting the inclined plane in B and T: and through E draw the vertical line Et meeting PB, GT, produced in b and t; also draw PQ perpendicular to EB. When the beam is in equilibrio, in the circumstances mentioned in the problem, the centre of gravity G will be at the highest point to which it can possibly arrive; that is, bt, or BT, which has a given ratio to bt, will be a maximum.

Put $HE = a$, $PQ = b$, $QB = d$, and $PE = x$; then $PG = a - x$, $EQ = \sqrt{(x^2 - b^2)}$, and $EB = \sqrt{(x^2 - b^2)} - d$; and by reason of the parallels PB, GT; $PE : EB :: PG : BT$

$$= \frac{PG \times EB}{PE} = \frac{a-x}{x} \times \left\{ \sqrt{(x^2 - b^2)} - d \right\} = \frac{a\sqrt{(x^2 - b^2)}}{x}$$

$$- \frac{ad}{x} = \sqrt{(x^2 - b^2)} + d$$
, which is to be a maximum: putting

its fluxion $= 0$, and reducing the resulting equation, gives $x^2 - ad\sqrt{(x^2 - b^2)} = ab^2$, from which equation x may be found.

When $d = 0$, that is, when the inclined plane becomes vertical, then $x^2 = ab^2$, the same as before.

Or, the problem may be resolved without fluxions.

Let HE (fig. 44, pl. 2.) represent the loaded rod; G its centre of gravity; EB the inclined plane, and the point P the prop: draw EF at right angles to EB, meeting GF perpendicular to the horizon in F, and join PF; when the rod is in equilibrio, PF will be perpendicular to GE. For the rod is then sustained by three forces, namely, the force of gravity in the direction GF; the reaction of the inclined plane EB in the direction EF perpendicular thereto; and the pressure against the prop P, which acts in a direction perpendicular to EG at P: and when the rod is in equilibrio, right lines in the said directions meet in the same point F.

Draw PQ perpendicular to EB, and draw PB parallel to the horizon, meeting Eb parallel to FG in b. Put $GE = a$, PQ

$=b$, $QB = d$, and $EP = x$; then $GP = -x$, $EQ = \sqrt{(a^2 - b^2)}$, and $EB = \sqrt{(a^2 - b^2)} - d$. The angles PQE , PbE , being right angles, a circle, whose diameter is EP , will pass through Q and b , and therefore $BP \cdot Bb = QB \cdot EB$; adding PB^2 to each, $PB^2 + BP \cdot Bb = PB \cdot Pb = PB^2 + QB \cdot EB$; whence $Pb = \frac{QB \cdot EB + PB^2}{PB}$; and by the similar triangles PBQ , EBb ;

$PB : PQ :: EB : Eb = \frac{PQ \cdot EB}{PB}$; also, by the similar triangles GPF , EBP , $GP : EF :: Eb = \frac{PQ \cdot EB}{PB}$; $Pb = \frac{BQ \cdot EB + PB^2}{PB}$,

whence $PQ \cdot PF \cdot EB = GP \times (QB \cdot EB + PB^2)$. Again, by the similar triangles PQE , EPF ; $PQ : EQ :: EP : PF$; whence $PQ \cdot PF = EP \cdot EQ$, by means of which the preceding expression becomes $EP \cdot EQ \cdot EB = GP \times (QB \cdot EB + PB^2)$; that is, $x \sqrt{(a^2 - b^2)} \times \{ \sqrt{(a^2 - b^2)} - d \} = (a - x) \times \{ d \sqrt{(a^2 - b^2)} + b^2 \}$, which equation, when properly reduced, becomes $x^2 - ad \sqrt{(a^2 - b^2)} = ab^2$, which is the very same conclusion as derived by the other method of solution.

It may be here observed, once for all, that the locus of the centre of gravity G is a nodated conchoid, having its axis in the line PQ ; this is sufficiently manifest from the generation of that curve.

XXX. PRIZE QUESTION 147, by LIMENUS.

Let A and B be two given points without a circle, given in magnitude and position; a point C may be found within the circle, such, that if any straight line be drawn through it, cutting the circle in F and G , and AF , BG , be joined, the square of AF will have to the square of BG , the ratio which is compounded of the ratio of FC to CG , and a certain given ratio, and which ratio is also to be found?

FIRST SOLUTION, by LIMENUS, the Proposer.

Conceive the point C (fig. 45, pl. 2.) to be determined, and let the circle which may pass through the points F , A , G , meet AC joined in E , then the rectangle $AC \cdot CE$ will be equal to $FC \cdot CG$, which is of a given magnitude, and thus E is a given point.

Also

Also, the included triangles ECC , FCA , will be similar, and EG will be to FA as EC to CF ; and, likewise, as CG to CA ; and therefore EG^2 will be to FA^2 in the ratio compounded of EC to CA , and GC to CF . But, by hypothesis, AF^2 is to PG^2 in the ratio compounded of FC to CG and a certain given ratio, which let be denoted by that of M to N . And, by compounding these two proportions, EG^2 will have to BG^2 the ratio compounded of EC to CA and M to N , which are given ratios. But E and B are given points, and EG and BG are lines drawn from thence to any point in the circumference of a given circle. Therefore those lines, or their squares, can only have a given ratio to one another in two circumstances, namely, when the points E , B , coincide, in which case, the ratio of EG^2 to BG^2 is that of equality; and when the other of these points E is so assumed that the given circle may satisfy, with respect to them, the well known problem of Apollonius on this occasion. For that purpose, the line joining B , E , must pass through the centre O of the given circle, and its radius must be a mean proportional to OB , OE ; for then EG^2 will always have to BG^2 the duplicate of the ratio in which EB is divided by the circumference, or that of OE to OB . From this we conclude that the point C must be situated in the line AB or in the line AE , and that it must be so taken in each case, that the points of intersection of any straight line drawn through it, with the given circle, shall be in the same circle, with the points A , E , or A , B , respectively, which are determinate problems.

Take OD in OA , a third proportional to it, and the radius OF ; draw BD to meet AE in C ; join OC and produce it to K , so that the radius OF be a mean proportional to OC , OK , and join KD and KE . As the rectangle $BO \cdot OE$ is equal to $AO \cdot OD$ and to $CO \cdot OK$, circles may pass through the points B , E , D , A , and B , E , C , K , and therefore the angle DAE is equal to DBE , and the latter of these is equal to CKE ; from which it follows that a circle is capable of passing through the points O , A , K , E , and that the rectangle $EC \cdot CA$ is consequently equal to $OC \cdot CK$. But it may be easily shewn that the rectangle $FC \cdot CG$ is equal to $OC \cdot CK$; it is therefore also equal to $AC \cdot CE$, and a circle is capable of passing through the points A , F , E , G , and, which, combined with what has been investigated above, demonstrates that the point C possesses the required property. And with respect to the constant ratio of M to N , it has been determined that that compounded with the ratio of EC to CA , must be the same with that of OE to OB ; it is therefore in the ratio compounded of AC to OB and of OE to EC , that is of AK to OK and OK to KB , which is the same with the ratio of AK to KB . But KO bisects the angle AKB , and therefore if AB be

drawn to meet OC in H , the ratio of M to N will be simply as AH to HB .

For the second case, draw DE to meet AB in C' , join OC' , and take OK' a third proportional to OC' and the radius OF , then by reasoning in the same manner, as in the former case, we shall perceive that the angle DAK' is equal to $DC'K'$, or $EC'K'$, which is equal to EBK' , and so the points O, A, B, K' , are in a circle, and the rectangle $AC' \cdot C'B$ is equal to $OC' \cdot C'K'$, which is equal to $FC' \cdot C'G'$. Thus a circle may pass through the points A, F', B, G' , and consequently C' is another point where the required property obtains. And from what has been determined, in this case, M will be to N as AC' to BC' . But it is obvious that the intersecting lines are so drawn as to determine CH an harmonical mean to CA, CB . Therefore this ratio of AC' to $C'B$ becomes identified with the former of AH to HB .

The construction will now be as follows. Find the centre O , of the given circle; join OA, OB , and therein find the points D, E , so that OD may be a mean proportional to OA and the radius, and OE a mean proportional to OB and the radius. Join AE, BD , to meet each other in C , through which draw any straight line to meet the circle in F and G ; then the square of AF will have to the square of BG the ratio which is compounded of FC to CG and AH to HB . Also draw DE to meet AB produced in C' , through which draw any straight line to meet the circle in F' and G' , and the square of AF' will have to the square of BG' the ratio which is compounded of FC' to $C'G'$ and AC' to $C'B$.

SECOND SOLUTION, by Mr. LOWRY.

Suppose that the point C (fig. 46, pl. 2.) is found, and that $a : b$ is the ratio which is to be determined. Then since the point C is given (by hyp.) the rectangle $GC \cdot CF$ is also given. On the line BC produced take the point D , such, that the rectangle $DC \cdot CB$ may be equal to the given rectangle $GC \cdot CF$: then D will be a given point, and BC, CD , will be given lines. Join DF ; then because the rectangles $GC \cdot CF$, and $DC \cdot CB$, are equal, the points B, F, D, G , are in a circle, and the triangles CDF, CGB , are similar;

therefore $BG : DF :: CG : CD$,

and $BG : DF :: BC : CF$;

therefore $BG^2 : DF^2 :: CG \cdot BC : CD \cdot CF$;

But, by hypothesis, $AF^2 : BG^2 :: CF \times a : CG \times b$.

Therefore $AF^2 : DF^2 :: BC \times a : CD \times b$ a given ratio: wherefore AF has to DF a given ratio; and A and D are given points, therefore the locus of the point F will be a circle given

given by position (Apol. 2, lib. 2nd.). But if this circle meets the given circle in F, then F will be a given point, which is contrary to the hypothesis. Therefore, in order that the point F may be indeterminate, and that the other conditions of the proposition may at the same time be fulfilled, it is necessary that the two circles coincide, or become identical.

When that is the case, the line joining the points A and D passes through the centre O, and the rectangle OD·OA is equal to the square of the semi-diameter of the circle (ibid): and since what has been shewn with regard to GF is true of any other chord passing through C, we have only to determine the point C, such, that the rectangle BC·CD may be equal to the rectangle GC·CF: hence this construction.

Draw AO to the centre of the circle, and take OD a third proportional to AO and the semi-diameter, and join BD. Assume any point F in the circumference (not in a straight line with the points B, D), and describe a circle through the points B, A, F, cutting the given circle in G; draw GF meeting BD in C: then C is the point sought.

To determine the ratio $a : b$; draw AI parallel to DC, and produce OC to meet AB in H, and AI in I: then H will be a given point, and AH, BH, given lines; and by the proposition referred to above

AF : DF :: OP (the radius) : OD, therefore
 $AF^2 : DF^2 :: BC \times a : CD \times b :: OP^2 : OD \cdot OA :: OD^2 : OA \cdot OD$,
 or, $BC \times a : CD \times b :: OA : OD :: AI : CD :: AI \times b : CD \times b$;
 therefore $BC \times a = AI \times b$, or $a : b :: AI : BC :: AH : BH$,
 the ratio required.

In the preceding analysis, the point C is within the circle, agreeably to the enunciation of the proposition: But a point c may be found without the circle, which will possess properties similar to the point within.

By assuming the point c as given, and proceeding exactly as in the former analysis, we find that AF has to DF a given ratio as before, which can only be the case when the ratio is that of equality, and then the points A and D coincide, and the points B, A, c, will be in a straight line. In this case $Bc \times a$ will be equal to $cA \times b$ ($CD \times b$), or $a : b :: cA : cB$; and the rectangle $cA \cdot cB$ equal to the rectangle $cf \cdot cg$. We have therefore to determine a point c in AB, such, that drawing any line from c to meet the circle in f and g, the rectangle $cf \cdot cg$ may be equal to the rectangle $cA \cdot cB$. This is done in the same manner as before, by assuming any point f in the circumference and describing a circle through the points A, B, f, to intersect the given circle in g, then lg being drawn to meet AB produced in c, it will be the point sought.

The

The point *C*, as determined above, is the same as the point *O*, in question 83, vol. 1, of the Repository. For, if *OE* be taken a third proportional to *BO* and the semi-diameter, the line joining the points *A*, *E*, will pass through *C*; also, if *ED* be drawn, it will pass through *c*. On *OC* produced, let *ON* be taken a third proportional to *OC* and the semi-diameter, and join *BN*, *ND*, *EC*, and *AC*; then it is obvious that the four points *A*, *D*, *C*, *N*; *N*, *D*, *O*, *B*; and *N*, *C*, *E*, *B*, are in the circumference of the same circles; therefore the angle $\angle ACN = \angle ADN = \angle ABO = \angle OCE$, and the points *E*, *C*, *A*, are in a straight line. Again, draw *cO* to the centre, and take *On* a third proportional to *Oc* and the semi-diameter, and join *An*, *Dn*, *ED*, and *Dc*; then the four points *A*, *D*, *n*, *c*; *A*, *D*, *E*, *B*; and *A*, *n*, *O*, *B*, are also in the circumference of the same circles; therefore the angle $\angle ADc = \angle Anc = \angle ABO = \angle EDO$, and the points *E*, *D*, *c*, are in the same straight line.

The trapezium *A*, *D*, *E*, *B*, being inscribed in a circle, the opposite sides meeting in *O* and *c*, and the diagonals in *C*; it is well known that $a : b :: AH : BH$, the same ratio as when the point *C* is within the circle.

It is obvious from this analysis that the given points may be either within or without the given circle, or one of them within and the other without. For if *OH* meet *ED* in *h*, it is evident, from comparing the above proportions, that

$$DF^2 : GE^2 :: CF \times Dh : CG \times Eh,$$

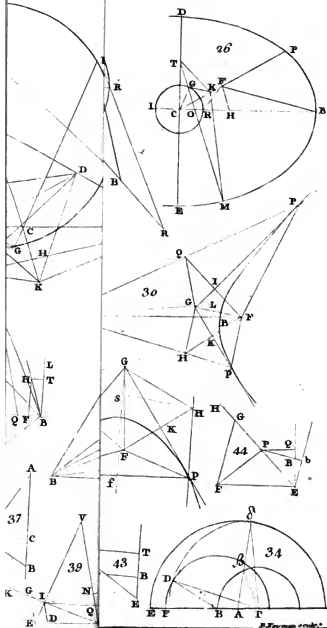
$$DF^2 : BG^2 :: CF \times CD : CG \times CB,$$

$$\text{and } AF^2 : GE^2 :: CF \times CE : CG \times CA.$$

When the points *A* and *B* are at the same distance from the centre, the line *EDc* will be parallel to *AB*, and in this case the point *C* only, can satisfy the conditions of the proposition.

The composition is so evident from the analysis that its insertion here is unnecessary.

LIMENUS is requested to send to Mr. GLENDINNING's for the Medal for solving the Prize Question.





NOTICES

RELATING TO MATHEMATICS.



I. DEATH OF THE BISHOP OF ST. ASAPH.

Since the publication of our last number, the Mathematical Sciences have lost an able and zealous cultivator, by the death of Dr. Samuel Horsley, Bishop of St. Asaph. This venerable prelate is well known to the world, as the author of various works, which have, no doubt, contributed greatly to the interests of learning as well as of religion and morality, but it is only of his mathematical writings that we can with propriety speak in this place. In 1769, he published a Restoration of the *Treatise of Inclinations of Apollonius*. Our readers no doubt know, that the original has long been lost. The historian of the mathematics, *Montucla*, has bestowed great commendation on this work; he says, that the treatise of the illustrious ancient has been restored "in the pure style, and with the peculiar elegance of the ancient geometry. It is probable, that, could Apollonius return again to the world he would not disavow it." Dr. Horsley gave to the world, in 1779, an elegant and complete edition of the writings of the immortal Newton, in five volumes quarto. He dedicated this work to the King, and illustrated it by a commentary. In 1801, the Bishop published an octavo volume, in English, entitled "Elementary Treatises, on the Fundamental Principles of Practical Mathematics, for the use of Students. Although published first, this is the last in order of three volumes of Elementary Geometry, which he sent forth from the Clarendon Press. The other two volumes are in Latin, the first contains the twelve books of Euclid's Elements; and the second, Euclid's Data; A book on the Properties of the Sphere; Archimedes on the Dimensions of the Circle; and Dr. Keil's Treatise on the Nature and Use of Logarithms. In 1805, he published a critical Essay on Virgil's two seasons of Honey, and his season for sowing Wheat, with a new and comprehensive Method of investigating the risings and settings of the fixed Stars. This was the last work he published relating to mathematics. He died in the 69th year of his age.

II. MATHEMATICAL WORKS IN THE PRESS.

Mr. Wallace of the R. M. College, will, in the course of the winter, publish a short Treatise on the Conic Sections.

A Second Edition of the first volume of a Course of Mathematics, designed for the use of the Officers and Cadets, of the Royal Military College by Isaac Dalby, Professor of Mathematics in the same College, will soon be published.

III. MATHEMATICAL WORKS LATELY PUBLISHED.

A Treatise on Plane and Spherical Trigonometry, with their most useful practical applications. By John Bonnycastle.

A Treatise on the Teeth of Wheels, Pinions, &c. demonstrating the best forms which can be given them for the various purposes of machinery, such as Clock-work, &c. from the French of M. Camus.

A complete collection of Tables for Navigation, and Nautical Astronomy; with simple, concise, and accurate methods for all the calculations useful at Sea. By I. de Mendoza Rios, Esq. F. R. S.

IV. FRENCH BOOKS LATELY IMPORTED.

Traité de Geodesie, ou Exposition des methodes Astronomiques et Trigonometriques Appliquees soit a la mesure de la Terre, soit a la confection du Canevas de Cartes et des Plans. Par L. Puissant, 4to. A Paris, 1805.

Traité Elementaire D'Astronomie Physique. Par I. B. Biot. 2 vols. 8vo.

Supplement au dixieme Livre du Traité de Mecanique Celeste sur L'Action Capillaire. 4to. p p. 65.

Leçons sur le Calcul des Fonctions, Nouvelle Edition, revue corrigee et augmentee par l'Auteur. (I. L. Lagrange.) 8vo. A Paris, 1806.

Note. This work is intended to serve as a Commentary and a Supplement to the first part of la Theorie des Fonctions Analytiques, published in 1796. It may be proper to observe that the work which we have here noticed was also published in 1803, as the 12th Cahier of Journal de L'Ecole Polytechnique.

ARTICLE II.

Solutions to Questions proposed in Number V.

I. QUESTION 151, by PAPPUS.

Given the length of a circular arc, to find the length of the chord connecting its extremities, so that the area of the included segment may be the greatest possible?

FIRST SOLUTION, by Mr. W. WALLACE, R. M. College.

As the area of the segment and the circular arch are each bisected by the diameter, it is evident that an answer to this question, and others of the same nature, may be easily derived from the solution of the following still more general problem.

Let *ade* (fig. 47, pl. 3.) be a curve of any kind whatever, given in species, and having a given position with respect to a straight line *ac*, to which it is referred as an axis; let *ADE* (fig. 48, pl. 3.) be another curve similar to *ade*, and *AC* its axis, and let the spaces contained between the curves and their axes be similar to one another. It is required to determine the co-ordinates *AB*, *BD* of *AD* an arch of the latter curve, when that arch is of a given magnitude, and the area *ABD* contained by the co-ordinates, and the arch is a maximum.

Let the arch *ad* of the curve *ade* be similar to the arch *AD*, and let *ab*, *bd* be its co-ordinates. Put *ab* = *x*, *bd* = *y*, the arch *ad* = *z*, the given arch *AD* = *a*, the area *ABD* = *v*; then, if we consider that $\int xy = \text{area } abd$, we have from similar figures,

$$z^2 : a^2 :: \int xy : v;$$

hence $v = \frac{a^2 \int xy}{z^2}$ a maximum,

and, consequently, taking the fluxions, &c.

$$\frac{\dot{v}}{x} = \frac{a^2}{z^2} \left\{ yz - \frac{\dot{z}}{z} \times z \int xy \right\} = 0;$$

from which we have this general formula,

$$yz = \frac{z}{x} \times 2 \int xy;$$

which is applicable, whatever be the nature of the curve.

If we suppose that ADE is a circle, as in the question, then *ade* being also a circle, let us suppose its radius unity, and put ϕ for the arch *ad*; then $z = \phi$, $x = 1 - \cos. \phi$, $y = \sin. \phi$, $\dot{z} = \dot{\phi}$, $\dot{x} = \dot{\phi} \sin. \phi$, and $2 \int xy = \phi - \cos. \phi \sin. \phi$; hence the general formula $yz = \frac{z}{x} \times 2 \int xy$ becomes in this particular case,

$$\phi \times \sin. \phi = \frac{1}{\sin. \phi} \times (\phi - \cos. \phi \sin. \phi),$$

from which we get,

$$\phi (1 - \sin.^2 \phi) - \cos. \phi \sin. \phi = 0,$$

that is,

$$\cos. \phi (\phi \cos. \phi - \sin. \phi) = 0.$$

Now there are just two ways in which we can satisfy this equation; we must put $\cos. \phi = 0$, or we must put $\phi \cos. \phi - \sin. \phi = 0$, which last equation is equivalent to $\phi = \tan. \phi$. If we put $\cos. \phi = 0$, then we have $\phi = 90^\circ$; and as to the equation $\phi = \tan. \phi$, it is evidently impossible, so long as ϕ is less than two right angles, as it must be by the nature of the question; therefore ϕ can only be a right angle, and the circular segment a semicircle, and this is the answer to the question.

For a second example, let us suppose the curve ADE, or the curve *ade*, to be a semi-parabola, and *ac* its axis. Suppose the parameter to be unity; then, because from the nature of the curve $\dot{z} = \dot{x} \sqrt{1 + \frac{1}{4x}}$, we have $\frac{\dot{z}}{\dot{x}} = \sqrt{1 + \frac{1}{4x}}$, and because $2 \int xy = \frac{2}{3}xy$, the general formula gives in this case

$$yz = \frac{2}{3}xy \times \sqrt{1 + \frac{1}{4x}},$$

$$\text{and } z = \frac{2}{3}x \sqrt{1 + \frac{1}{4x}} = \frac{2}{3} \sqrt{4x^2 + x^3}.$$

Now if *dt* be drawn touching the curve, and meeting the axis in *d*, it is evident, from the nature of the curve, that the tangent *dt* is equal to $\sqrt{4x^2 + x^3}$; therefore in the case of the parabola the arch *ad* = $\frac{2}{3}$ tangent *dt*. And since the same must be true of the curve ADE, if *DT* be drawn touching it and meeting the axis in *T*, we must also have *AD* = $\frac{2}{3}$ *DT*; and since *AD* is by hypothesis given, therefore *DT* is given.

Put

Put ϕ for the angle T , and p for the parameter; then, from the nature of the curve,

$$2 \text{ tangent } DT = p \times \cot. \phi \operatorname{cosec}. \phi,$$

$$\text{arch } AD = \frac{p}{4} \left\{ \cot. \phi \operatorname{cosec}. \phi + \text{hyp. log. } (\cot. \phi + \operatorname{cosec}. \phi) \right\}.$$

From these equations, and because $\text{arch } AD = \frac{1}{3} \text{ tangent } DT$, we find

$$\cot. \phi \operatorname{cosec}. \phi = 3 \text{ hyp. log. } (\cot. \phi + \operatorname{cosec}. \phi);$$

hence the angle ϕ may be easily determined by the Trigonometrical Tables, and the method of Trial and Error. Now the angle ϕ , and the line DT being known, the parameter of the parabola, and every thing else required, may be readily found.

SECOND SOLUTION, by Mr. IVORY, R. M. College.

Let a denote the given length of the arc, and let s denote the area of the corresponding segment when that area is a maximum, or greater than the area corresponding to an arc of equal length in any other circle. Let z denote the area of the segment, that is similar to the segment s , in a circle described with any assumed radius (as with the radius 1), and let u be the length of the corresponding arch: then because the segments s and z are similar, therefore, as s is to z , so is a^2 to u^2 ; and consequently,

$$s = \frac{z}{u^2} \times a^2.$$

Let π denote half the circumference, whose radius is unity; then, supposing u to be less than π , $z = \frac{u - \sin. u}{2}$, and $2s =$

$$\frac{\pi (u - \sin. u)}{u^2} \times \frac{a^2}{\pi}; \text{ because an arc is less than its tangent, therefore}$$

$$\frac{2 \sin. \frac{u}{2}}{\cos. \frac{u}{2}} > u; \text{ and } 2 \sin. \frac{u}{2} > u \cos. \frac{u}{2}; \text{ and } 2 \sin. \frac{u}{2} \cos. \frac{u}{2}, \text{ or}$$

$$\sin. u, > u \cos. \frac{u}{2}; \text{ and } u \sin. \frac{u}{2} > u - \sin. u. \text{ But the tan-}$$

$$\text{gent of the arc } \frac{\pi - u}{2} \text{ is } \frac{\cos. \frac{u}{2}}{\sin. \frac{u}{2}}; \text{ therefore } \frac{2 \cos. \frac{u}{2}}{\sin. \frac{u}{2}} > \pi - u.$$

Therefore $u \sin. \frac{u}{2} \times \frac{2 \cos. \frac{u}{2}}{\sin. \frac{u}{2}} > (\pi - u) \times (u - \sin. u)$, that

is, $u \times \sin. \frac{u}{2} \times 2 \cos. \frac{u}{2}$, or $u \sin. u, > \pi u - u^2 - \pi \sin. u + u \sin. u$; and adding $u^2 - u \sin. u$ to the unequal quantities, $u^2 > \pi \times (u - \sin. u)$: Therefore when u is less than π , then $2s \left(= \frac{\pi(u - \sin. u)}{u^2} \times \frac{a^2}{\pi} \right)$ is less than $\frac{a^2}{\pi}$.

And, if u be supposed greater than π , then $z = \frac{u + \sin. u}{2}$, and $2s = \frac{\pi(u + \sin. u)}{u^2} \times \frac{a^2}{\pi}$. Let $u = \pi + w$, then, $\sin. u = \sin. w$, and $2s = \frac{\pi(\pi + w + \sin. w)}{(\pi + w)^2} \times \frac{a^2}{\pi}$; but $\pi + w + \sin. w < \pi + 2w$, therefore $2s < \frac{\pi(\pi + 2w)}{(\pi + w)^2} \times \frac{a^2}{\pi}$. Now $\pi(\pi + 2w)$ is less than $(\pi + w)^2 = \pi(\pi + 2w) + w^2$; therefore when u is greater than π , $2s$ is less than $\frac{a^2}{\pi}$.

But when $u = \pi$, then $2s = \frac{a^2}{\pi}$; and it has been shewn that $2s$ is less than $\frac{a^2}{\pi}$, both when u is less than π , and when it is greater than π : Therefore s is greatest of all when u is = a semicircle. But the segment s is similar to the segment z ; therefore of all the circular segments, that have the same length of arc, the semicircle is the greatest.

SCHOLIUM.

This curious proposition is taken from the 5th Book of the Mathematical Collections of Pappus (Prop. 10), and we have done little more than translate the geometrical demonstration given there into algebraic language. The demonstration is the more worthy of notice that it is founded on a principle which, although it remained sterile in the hands of the ancient geometicians, became fertile in consequences on the discovery of the modern methods of Maxima and Minima. If u and a denote
similar

similar arcs of two similar curves, and z and s be the corresponding areas, then because the same reasoning that is applied to similar segments of circles in the foregoing demonstration will equally hold of any similar portions of similar curves, therefore

$s = \frac{z}{u^2} \times a^2$. If now, the arch a , be supposed to be invariable, or of a given length, then it is manifest that the area s will

be a maximum when $\frac{z}{u^2}$ is a maximum, that is by taking flux-

ions, when $u \times dz - 2z \times du = 0$. Let x be the absciss and y the rectangular co-ordinate of the area z , then $dz = ydx$, and

$du = dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$: therefore, by substitution, $u =$

$\frac{\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{y} \times 2fydx$. Therefore, "if in any curve what-

ever an arch u be taken so as to satisfy the equation $u =$

$\frac{\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{y} \times 2fydx$, and if another curve be found simi-

lar to the proposed curve, such, that the arch of the second curve, which is similar to the arch u of the first curve, shall be of a given length a : then the area of the second curve that corresponds to the arch a will be a maximum, that is, it will be greater than the area that corresponds to an arch of the given length a in any other curve similar to the proposed curve."

In applying this general solution to the problem of Pappus, it is to be remarked that z and u are here supposed to denote the halves of which they were made to stand for in the demonstration above, that is, u denotes half the length of the arch of the required segment, and z denotes half the area of it: then $y = \sin.$

u , and $x = 1 - \cos. u$, and $\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} = \sqrt{\left(1 + \frac{\cos.^2 u}{\sin.^2 u}\right)}$

$= \frac{1}{\sin. u}$: therefore the formula gives $u = \frac{2fdu \times \sin.^2 u}{\sin.^2 u}$

Hence $u \sin.^2 u = 2fdu \times \sin.^2 u = fdu - fdu \times \cos. 2u$
 $= u - \frac{1}{2} \sin. 2u$: and $u \cos.^2 u = \frac{1}{2} \sin. 2u$, an equation which it is easy to prove can not take place unless when $\cos. u$, and $\sin. 2u$ are both $= 0$, that is when u is a quadrant, and the whole segment is a semi-circle.

If

If the proposed curve be a parabola, whose equation is $4px = y^2$, then the formula gives $u = \frac{\sqrt{(4p^2 + y^2)}}{y} \times \frac{1}{3} \times \frac{y^3}{p} = \frac{1}{3} \cdot \frac{y\sqrt{(4p^2 + y^2)}}{p}$, but $u = \frac{y\sqrt{(4p^2 + y^2)}}{4p} + p \times \log. \frac{y + \sqrt{(4p^2 + y^2)}}{2p}$ (Simpson's fluxions, vol. 1, page 162); therefore, by substitution, we get $p \times \log. \frac{y + \sqrt{(4p^2 + y^2)}}{2p} = \frac{y\sqrt{(4p^2 + y^2)}}{12p}$, which equation defines the portion of the curve on which the maximum depends in the case of the parabola.

But the principle on which the demonstration of Pappus is founded, admits of much more extensive application than to the solution of the particular problem we have been considering. For if $f(x, y)$, and $F(x, y)$ denote any functions whatever of the co-ordinates of a curve, and it is required to determine a curve similar to the proposed curve, such that $f(x, y)$ shall be a maximum or minimum, when $F(x, y)$ is of a given magnitude: then we may resolve this very general problem by reasoning in the same manner as is done above. Let x' and y' denote the co-ordinates of the curve to be found that are similar to the co-ordinates x and y of the proposed curve, and let m denote the dimensions of the function $f(x, y)$. (The nature of such questions requires that the functions $f(x, y)$ and $F(x, y)$ be homogeneous when the dimensions of the co-ordinates and the parameters, or constant lines that depend on the curve, are reckoned), and let n denote the dimensions of the function $F(x, y)$: then because x, y , and x', y' are similar co-ordinates of similar curves, by reducing the functions to homogeneous dimensions, we shall get

$$\{f(x, y)\}^n : \{F(x, y)\}^m :: \{f(x', y')\}^n : \{F(x', y')\}^m$$

$$\text{whence } f(x', y') = \frac{f(x, y)}{\{F(x, y)\}^{\frac{m}{n}}} \times \{F(x', y')\}^{\frac{m}{n}}$$

and because $F(x', y')$ is to be of a given magnitude, therefore

$$f(x', y') \text{ will be a maximum or minimum, when } \frac{f(x, y)}{\{F(x, y)\}^{\frac{m}{n}}} \text{ is}$$

is a maximum or minimum. And thus the problem is made to depend on a case that comes under the common method of maxima and minima.

For the sake of illustration let the following problem be proposed: Let A (fig. 49, pl. 3.) be a given point in a given horizontal line AD, and BC a line given by position in the same vertical plane; it is required to determine a cycloid, that shall pass through A and have its base in the line AD, such, that a heavy body shall descend from the point A to the line BC, in that cycloid, in a less time than in any other cycloid passing through the point A and having its base in the line AD.

Let AD be half the base of the cycloid to be determined, and DE the diameter of its generating circle: Let PQR (fig. 50, pl. 3.) be the semi-cycloid of which the radius of the generating circle is 1, and let PL be the like arch of the semi-cycloid PR, that AC is of the semi-cycloid AE, and draw the line LM similarly to the line CB: draw LN, LK parallel to QR and PQ, and let LK cut the generating circle in H: let π = the semi-circumference QHR, ϕ = arch QH, and T = the whole time of descent in the semi-cycloid PLR:

then the time of descent in the arch PL = $\frac{\phi}{\pi} \times T$: and the time of descent in the arch AC, similar to the arch PL, = $\frac{\phi}{\pi} \times T \times \frac{\sqrt{(DE)}}{\sqrt{(QR)}}$: but because the lines BC and ML cut the two cycloids similarly, therefore $\frac{DE}{QR} = \frac{AB}{PM}$: therefore the time of descent in the arch AC = $\frac{\phi}{\pi} \times T \times \frac{\sqrt{(AB)}}{\sqrt{(PM)}}$, which will be a minimum when $\frac{\phi}{\sqrt{(PM)}}$, or $\frac{\text{arch QH}}{\sqrt{(PM)}}$, is a minimum, all the other quantities being constant.

I hardly need to remark that $\frac{\text{arch QH}}{\sqrt{(PM)}}$ is what the general formula $\frac{f(x, y)}{\{F(x, y)\}^{\frac{m}{n}}}$ becomes in the particular case under consideration.

Let the cotangent of the angle ABC, or PML, be = c ; then NL = QK = $1 - \cos. \phi$; and MN = $c \times NL = c - c \times \cos.$

$\propto \cos. \varphi$: also, from the nature of the cycloid $PN = \text{arch } QH - KH = \varphi - \sin. \varphi$:

therefore $PM = PN + NM = \varphi - \sin. \varphi + c - c \times \cos. \varphi$.

Therefore $\frac{\varphi}{\sqrt{PM}} = \frac{\varphi}{\sqrt{(\varphi - \sin. \varphi + c - c \times \cos. \varphi)}}$, which

expression is to be a minimum. By taking fluxions, dividing by $d\varphi$, and throwing away the denominator, there results,

$$\varphi - \sin. \varphi + c - c \times \cos. \varphi - \frac{1}{2} \times \varphi (1 - \cos. \varphi + c \times \sin. \varphi) = 0.$$

$$\text{Hence } \varphi \times \{1 + \cos. \varphi - c \times \sin. \varphi\} = 2 \{ \sin. \varphi - c \times (1 - \cos. \varphi) \} :$$

but $1 - \cos. \varphi = \frac{\sin.^2 \varphi}{1 + \cos. \varphi}$, therefore, by substitution,

$$\varphi \times (1 + \cos. \varphi) \times \{1 + \cos. \varphi - c \times \sin. \varphi\} = 2 \sin. \varphi \times \{1 + \cos. \varphi - c \times \sin. \varphi\}$$

$\varphi - c \times \sin. \varphi$ } an equation that cannot take place unless

$1 + \cos. \varphi = c \times \sin. \varphi$. Now $KR = 1 + \cos. \varphi$ and $KH = \sin. \varphi$, therefore c is the tangent of the angle QRH , and thus we conclude that the chord QH is parallel to LM , and that $PM = \text{arc } QH$, and $QM = \text{arch } HR = HL$. Hence this construction, "Make the angle $PQH = \text{the angle } ABC$, and as arch QH to arch HR so make AB to BD ; then AD will be the base of the semi-cycloid required."

James Bernoulli seems to be the first that, in modern times, observed the importance of this principle, drawn from the comparison of similar curves, in the solution of problems relating to maxima and minima. He took the occasion of his discovery to propose many problems, relating to circles, parabolas, and cycloids, to his brother John Bernoulli and other rival mathematicians; always taking care to choose such instances as could be enunciated without any mention of the similarity of the curves under consideration, that he might not betray the source whence he drew his solutions (his words are, *ne fundamentum solutionis nimis aperte darem*).

II. QUESTION 152, by HERMODORUS.

To inscribe seven equal circles in a given circle?

SOLUTION, by Mr. JOHN CAVILL, Beighton.

Divide the circumference of the given circle into six equal parts in the points A, B, C, D, E and F, (fig. 51, pl. 3.); then the lines

lines AD, BE, and CF which join the opposite points of division; will pass through G, the centre of the circle. With a distance equal to a third part of the given radius and centre G, describe a circle to intersect the three diameters in the points *a, b, c, d, e, f*; then this will be one of the inscribed circles, and A*a*, B*b*, C*c*, D*d*, E*e*, F*f*, will be the diameters of the six other circles.

For since the radius of each of the inscribed circles is a third part of the radius of the given circle, those circles must be equal, and the exterior and interior circles will evidently touch the other six; we have therefore only to prove that the circles described on A*a*, B*b*, C*c*, &c. touch each other.

Let *m* and *n* be the centres of any two contiguous circles, and join *mn*; then since AB is a sixth part of the circumference, the chord AB is equal to the radius AG or GB, and therefore, *mn* is equal to *mG* or *nG*, that is *mn* is equal to *ma* + *nb*, or to the sum of the radii of the circles described on the diameters A*a*, B*b*; and, therefore, these circles touch each other in the middle of the line *mn*. And it is proved in the same manner, that the remaining circles touch each other; and, therefore, seven equal circles are inscribed in the given circle as required.

Solutions were also received from Messrs. Swale and Toplis.

III. QUESTION 153, by CENTROBARICOS.

The radius of the greater end of the frustum of a paraboloid being R, that of its less end *r*, and the height of the frustum *h*; it is required to give a neat expression for the position of its centre of gravity?

FIRST SOLUTION, by the Rev. JOHN TOPLIS, Arnold, Notts.

Imagine the frustum of the paraboloid to be completed, and let *a* denote the part of the axis intercepted between the end (or ordinate) R and the vertex; also let *x* denote the thickness of the frustum from the end R, the position of whose centre of gravity is required. Then by the properties of the parabola and centre of gravity,

$$\frac{\int x \dot{x}(a-x)}{\int \dot{x}(a-x)} = \frac{\frac{1}{2}ax^2 - \frac{1}{3}x^3}{ax - \frac{1}{2}x^2} = \frac{3ax - x^2}{6a - 3x}$$

will express the distance of the centre of gravity from the end R, measured along the axis.

Again by the property of the parabola, $a = \frac{R^2 x}{R^2 - r^2}$, and by means of this equation, exterminating a from the preceding expression for the distance of the centre of gravity from the end R , $\frac{3ax - 2x^2}{6a - 3x} = \frac{3R^2 x^2 \div (R^2 - r^2) - 2x^2}{6R^2 x \div (R^2 - r^2) - 3x} = \frac{x(R^2 + 2r^2)}{3(R^2 + r^2)}$ and writing h for x the expression will be $\frac{h(R^2 + 2r^2)}{3(R^2 + r^2)}$ which is what was required.

SECOND SOLUTION, by Mr. JOHNSON, Birmingham.

Let BDCH (fig. 52. pl. 3.) be the given frustrum, A the vertex, and AIP the axis of the paraboloid when completed. Put PH = R, IC = r , IP = h , $n = .78539$, &c. and AI = x . Take EI = $\frac{1}{2}$ AI and PF = $\frac{1}{2}$ AP; then EF = $\frac{1}{2}$ IP = $\frac{1}{2}h$, and E is the centre of gravity of the conoid ADC, and F the centre of gravity of the conoid BAH.

Let G be the required centre of gravity of the frustrum BDCH; then because the solid is symmetrical to the axis IP, the point G must be in that axis; and by the nature of the centre of gravity, EF = $\frac{1}{2}h$: FG :: the solidity of the frustrum BDCH = $2nh(R^2 + r^2)$: the solidity of the conoid ADC = $2\pi x r^2$; therefore

$$FG = \frac{1}{2}x \left(\frac{r^2}{R^2 + r^2} \right), \text{ and } IG = \frac{1}{2}h - \frac{1}{2}x + \frac{1}{2}x \left(\frac{r^2}{R^2 + r^2} \right) \\ = \frac{1}{2} \left\{ \frac{h(R^2 + r^2) - \frac{1}{2}x(R^2 - r^2)}{R^2 + r^2} \right\}.$$

But by the property of the parabola $x : x + h :: r^2 : R^2$, or $x = \frac{h \times r^2}{R^2 - r^2}$; this value of x being substituted in the expression for IG, we have $\frac{h}{3} \times \left(\frac{2R^2 + r^2}{R^2 + r^2} \right)$ for the distance of the centre of gravity, from the left end of the frustrum.

Mr. Whitley also answered it.

IV. QUESTION 154, by DIOPHANTUS.

Find a trapezium which can be inscribed in a circle, such, that its sides and area may be expressed by whole numbers?

SOLUTION,

SOLUTION, by Mr. CUNLIFFE, R. M. College.

Let v, x, y and z denote the sides of the trapezium that can be inscribed in a circle; then by a known theorem $(v+x+y-z)$ $(v+x+z-y)$ $(v+y+z-x)$ $(x+y+z-v)$ will express 16 times the square of its area, which must therefore be a square number, because 16 is a square number.

Put $v+x+y-z = \frac{ma}{n}$, $v+x+z-y = \frac{na}{m}$, $v+y+z-x = \frac{rb}{s}$, and $x+y+z-v = \frac{sb}{r}$; half the sum of the four equations is

$$v+x+y+z = \frac{a(m^2+n^2)}{2mn} + \frac{b(r^2+s^2)}{2rs}.$$

Again taking half the difference between this expression and each of the assumed equations

$$z = \frac{a \times (n^2 - m^2)}{4mn} + \frac{b \times (r^2 + s^2)}{4rs}; \quad y = \frac{a \times (m^2 - n^2)}{4mn} + \frac{b \times (r^2 + s^2)}{4rs};$$

$$x = \frac{a \times (m^2 + n^2)}{4mn} + \frac{b \times (s^2 - r^2)}{4rs}; \quad v = \frac{a \times (m^2 + n^2)}{4mn} + \frac{b \times (r^2 - s^2)}{4rs}.$$

If it is desired to have expressions for the sides in whole numbers these may be had by taking $a = 4mn$ and $b = 4rs$, and then $z = n^2 + r^2 + s^2 - m^2$, $y = m^2 + r^2 + s^2 - n^2$, $x = m^2 + n^2 + s^2 - r^2$, $v = m^2 + n^2 + r^2 - s^2$, where m, n, r and s may be taken at pleasure.

Take $m = 5$, $n = 4$, $r = 3$, and $s = 2$; then $z = 4$, $y = 22$, $x = 36$, and $v = 46$, but each of these is divisible by 2; therefore we may take $z = 2$, $y = 11$, $x = 18$, $v = 23$, which are the numerical values of four sides of a trapezium that may be inscribed in a circle and have its area expressible by rational whole numbers.

Ingenious solutions were also received from Messrs. Swale and Witley.

V. QUESTION 155, by Mr. J. JOHNSON.

In a given spherical triangle, it is required to inscribe another, so that its perimeter may be the least possible?

FIRST SOLUTION, by Mr. WALLACE.

Let ABC (fig. 53, pl. 3,) be the given triangle, and suppose PQR to be the required inscribed triangle. Now if these in-

stead of being spherical, were plane, it is easy to see by a well known mode of reasoning that any two sides, as PR, QR of the inscribed triangle must make equal angles PRB, QRC at the point in which they meet BC a side of the circumscribed triangle; and in like manner it may be shewn that the same must be true when the triangles are spherical. Produce RP, RQ, and from A draw Ad, Ae. arches of great circles perpendicular to RP, RQ, and Af perpendicular to PQ: Then, because the angle APQ, or APf is equal to RPB, that is to APd, it follows that the arch Af is equal to the arch Ad; and in like manner it appears that the arch Ae is equal to Af; thus the arches Ad, Ae are equal. Therefore, if a great circle be drawn through A and R, the angle ARd is equal to ARe; but the angle PRB has been found equal to QRC, therefore the angle ARB is equal to ARC, that is, AR is perpendicular to BC. In the very same way it may be shewn that the points P and Q are determined by drawing arches of great circles from B and C (the remaining angles of the given triangle) perpendicular to the opposite sides AB, AC; hence the solution of the question is manifest.

SECOND SOLUTION, by Mr. JOHNSON, the Proposer.

Let PQR (fig. 54, pl. 3.) be a triangle inscribed in the given spherical triangle ACB, and having its perimeter the least possible. Then, by reasoning, as in the solution to question 120, it will appear that the perimeter of the inscribed triangle is a minimum, when the angle CQR is equal to the angle AQP, the angle APQ equal to the angle BPR, and the angle BRP equal to the angle CRQ.

To determine the position of the points P, Q, R, let the arc CP be drawn to meet QR in I; then the angles CQP, CRP, being the supplements of the angles CQR, CRQ, we have by trigonometry

$$\sin CQ : \sin CP :: \sin CPQ : \sin PQC = \sin CQI$$

$$\sin IC : \sin CQ :: \sin CQI : \sin CIQ$$

therefore $\sin IC : \sin CP :: \sin CPQ : \sin CIQ$.

It is shewn in the same manner

that $\sin IC : \sin CP :: \sin CPR : \sin CIR$

therefore $\sin CPQ : \sin CIQ :: \sin CPR : \sin CIR$;

but $\sin CIQ = \sin CIR$, therefore $\sin CPQ = \sin CPR$, or the angle QPC is equal to the angle CPR; but the angles APQ, BPR are also equal, therefore the angle APC is equal to the angle BPC, or CP is perpendicular to AB. In like manner it is proved that the arc AR is perpendicular to CB, and the arc BQ perpendicular to AC, therefore the points P, Q, R are found by drawing perpendiculars from the angular points A, C, B to the opposite sides.

VI. QUESTION

VI. QUESTION 156, *by* LIMENUS.

If a straight line be drawn to meet the four sides of a trapezium in a, b, c, d , and the two diagonals in m and n ; then will ab be to cd as $am \times nb$ to $cm \times nd$. Required the demonstration?

SOLUTION, *by* AMICUS.

Fig. 55, pl. 3. Through one of the angular points of the trapezium, as C , draw a straight line parallel to $abcd$, meeting the sides AB, AC in Q and I , and the diagonal BD in P .

Then by reason of the parallels

$$ab : QC :: bn : CP$$

$$CI : cd :: CP : nd$$

$$QC : CI :: am : mc;$$

therefore by composition of ratios

$$ab : cd :: am \cdot bn : mc \cdot nd.$$

VII. QUESTION 157, *by* J. W. LEEDS.

Let A be a given point in the periphery of a given circle, let any chord AC be drawn, and divided in the point E , so that the rectangle $AC \times CE$ shall be equal to a given space; it is required to determine the equation and quadrature of the locus of the point E ?

SOLUTION, *by* Mr. SWALE.

Let AB (fig. 56, pl. 3,) be the diameter of the circle, and take the point Q such, that $AB \cdot BQ$ may be equal to the given rectangle $AC \cdot CE$; draw EP perpendicular to AB and join BC . Put $AB = a$, $BQ = b$, $AQ = c$, any abscissa $QP = x$, and the corresponding ordinate $PE = y$.

Then $AE = \sqrt{\{(c-x)^2 + y^2\}}$ and by similar triangles

$$AE : AP :: AB : AC = \frac{a(c-x)}{\sqrt{\{(c-x)^2 + y^2\}}}; \text{ and}$$

$$AC \cdot CE = AC^2 - AC \cdot AE = AC^2 - AB \cdot AP = ab;$$

therefore $\frac{a^2(c-x)^2}{(c-x)^2 + y^2} - a(c-x) = ab$; and by reduction we have

have $y = \pm (c - x) \sqrt{\frac{x}{a - x}}$ the equation of the curve.

When y is $= 0$, then x is either $= 0$, or to c ; therefore the curve passes through the points Q , A ; and if the ordinates be taken on the other side of AQ the curve will again pass through A , therefore A is a *punctum duplex*. Again, if x be greater than c , or P be on BA produced to the left of A , then y

is $= \pm (x - c) \sqrt{\frac{x}{a - x}}$, where, if x is equal to a , then

y is infinite, which shews that if QK be taken $= a$, then SKS drawn perpendicular to BK will be an asymptote to the curve.

When the point Q is between the points A , B , or the given rectangle is less than the square of the diameter, the curve has a *nodus* between A and B .

But when the points A and Q coincide, or the given rectangle is equal to the square of the diameter, then $c = 0$, and

$y = \pm x \sqrt{\frac{x}{a - x}}$, which is an equation to the cissoid of the

ancients. In this case the *nodus* vanishes, and the curve has a *cusps* at A , the two arcs of the curve touching one another in that point.

If the point Q be on the left of A , or the given rectangle be greater than the square of the diameter, then y is $= \pm$

$(c + x) \sqrt{\frac{x}{a - x}}$, in which equation, if y is $= 0$, then x is

also $= 0$; and if x is $= a$, then y is infinite, therefore the curve passes through Q' , and taking $Q'K' = a$, a line drawn through K' perpendicular to AK' will be an asymptote to the curve, and the *nodus* will in this case become a conjugate point at A .

Lastly, if we suppose the point Q'' to be on the right of the point B , then the given rectangle may be either equal to, or greater or less than the square of the diameter, and the general equation of the curve will be $y = \pm (c - x) \sqrt{\frac{x}{a - x}}$, where y is $= 0$

if $x = c$, and y infinite when $a = x$; hence the curve passes through Q'' , and if AK'' be $= BQ''$, the line $S''K''S''$ perpendicular to AQ will be an asymptote to the curve.

To find the area we have

$$y \dot{x} = \dot{x} (c - x) \sqrt{\frac{x}{a - x}} = \frac{cx\dot{x}}{\sqrt{(ax - x^2)}} - \frac{x^2\dot{x}}{\sqrt{(ax - x^2)}}.$$

Now

Now $\int \frac{cx\dot{x}}{\sqrt{(ax-x^2)}}$ is $= c(z-w)$, where z is the circular arc to radius a and verfed line x , and $w = \sqrt{(ax-x^2)}$. Also $\int \frac{x^2\dot{x}}{\sqrt{(ax-x^2)}} = \frac{1}{2} \int \frac{ax\dot{x}}{\sqrt{(ax-x^2)}} - \frac{1}{2}xw = \frac{1}{2}az - (\frac{1}{2}x + \frac{1}{2}a)w$, and $\int y\dot{x} = (c - \frac{1}{2}a)z + (\frac{1}{2}a + \frac{1}{2}x - c)w =$ (when $x=c$) to $(c - \frac{1}{2}a)z + (\frac{1}{2}a - \frac{1}{2}c)\sqrt{(ac-c^2)} =$ to the whole area of the semi-nodus AEQ.

Again, the fluxion of the area Aep is $= \frac{x^2\dot{x}}{\sqrt{(ax-x^2)}} - \frac{cx\dot{x}}{\sqrt{(ax-x^2)}}$ and $\int y\dot{x} = (\frac{1}{2}a - c)z + (\frac{1}{2}x - \frac{1}{2}a - c)w$ which ought to vanish when $x=c$, but it is then $= (\frac{1}{2}a - c)z' - (\frac{1}{2}a + \frac{1}{2}c)w'$, where z' is the circular arc whose radius is $\frac{1}{2}a$, and verfed line c , and $w' = \sqrt{(ac-c^2)}$; wherefore the correct fluent is $(\frac{1}{2}a - c)(z - z') + (\frac{1}{2}x - \frac{1}{2}a - c)w - (\frac{1}{2}a + \frac{1}{2}c)w' =$ (when $x=a$) to $(\frac{1}{2}a - c)(z - z') + (c - \frac{1}{2}a)w - (\frac{1}{2}a + \frac{1}{2}c)w'$ the area of the space contained between the curve and the asymptote.

Ingenious solutions were likewise received from Messrs. Cavill, Nicholson, Toplis, and Whitley.

VIII. QUESTION 158, by JUVENIS.

Given $2x\dot{x} - 3y\dot{y} = 10\dot{y} - 5x\dot{y}$, to find the equation of the fluents?

FIRST SOLUTION, by the proposer, JUVENIS.

In the given equation substitute $u+2$ for x , and $v+\frac{1}{2}$ for y , (which values are found by M^DAM AGNEST'S *Analytical Institutions*, Book^t iv, Sect. ii, Art. 22,) and we shall have

$$(2u+4-3v-4) \times \dot{u} = (10-5u-10) \times \dot{v},$$

that is, $2u\dot{u} - 3v\dot{v} = -5u\dot{v}$, or $2u\dot{u} = 3v\dot{v} - 5u\dot{v}$.

Now this equation is of the form of one of those treated of in Art. 7, Sect. i, of the aforesaid book, and may be solved as follows:

First change the signs, and it becomes

$$-2u\dot{u} = 5u\dot{v} - 3v\dot{v}.$$

Secondly,

Secondly, multiply it by $u^{-\frac{2}{3}}$, and there will be

$$-2\dot{u}u^{\frac{1}{3}} = 5\dot{v}u^{\frac{2}{3}} - 3v\dot{u}u^{-\frac{1}{3}}.$$

Thirdly, divide this equation by $25u^{\frac{6}{5}}$, and we shall have

$$\frac{-2\dot{u}u^{-\frac{1}{3}}}{25} = \frac{5\dot{v}u^{\frac{2}{3}} - 3v\dot{u}u^{-\frac{1}{3}}}{25u^{\frac{6}{5}}};$$

from which the correct equation of the fluents is evidently

$$\frac{c - u^{\frac{5}{3}}}{5} = \frac{v}{5u^{\frac{3}{5}}}, \text{ or, } 5cu^{\frac{2}{3}} - u = v.$$

And by restoring the values of x and y , this equation becomes

$$5c \times (x - 2)^{\frac{2}{3}} - x + 2 = y - \frac{1}{3};$$

that is, $5c \times (x - 2)^{\frac{2}{3}} - x + \frac{1}{3} = y$, the equation required.

Remark. If c be put $= 0$, the *locus* of the equation will be a line of the first order, that is a right line; if c be any constant quantity, the *locus* of the equation will be a line of the fifth order. This is a remarkable instance of the truth of what *Madam Agnesi* has said in her *Anal. Instit.* Book iv, Sect. ii, Art. 13.

SECOND SOLUTION, by ———.

To prepare the fluxional equation for a solution, let it be put under this form

$$\frac{3y}{10 - 5x} + \frac{\dot{y}}{x} = \frac{2x}{10 - 5x},$$

this is evidently a particular case of the more general equation

$$Ay + \frac{\dot{y}}{x} = X,$$

where A and X denote any functions whatever of the variable quantity x . Writers on Analysis have employed different artifices to separate the variable quantities x and y in this equation. The following method, perhaps the most elegant of any, is given by Lagrange in his *Theorie des Fonctions Analytique*, page 62.

Let p be such a value of y as satisfies the equation

$$Ay + \frac{\dot{y}}{x} = 0;$$

so that we have $Ap + \frac{\dot{p}}{x} = 0$. Suppose now that $y = ap$, where a denotes a function of x to be presently determined. Then, taking the fluxions, we have $\dot{y} = a\dot{p} + p\dot{a}$ and $\frac{\dot{y}}{x} = \frac{a\dot{p}}{x} + \frac{p\dot{a}}{x}$. Let the values of y and $\frac{\dot{y}}{x}$ be now substituted in the equation $Ay + \frac{\dot{y}}{x} = X$, and it becomes

$$a \left(Ap + \frac{\dot{p}}{x} \right) + \frac{p\dot{a}}{x} = X.$$

But $Ap + \frac{\dot{p}}{x} = 0$, therefore $\frac{p\dot{a}}{x} = X$, and $a = \int \frac{X\dot{x}}{p} + c$, putting c for a constant quantity.

Now from the equation $Ap + \frac{\dot{p}}{x} = 0$, we have $\frac{\dot{p}}{p} = -$

$A\dot{x}$, and $p = e^{-\int A\dot{x}}$, (where e is the number whose hyperbolic log. = 1). Thus we have upon the whole,

$$y = pa = e^{-\int A\dot{x}} \left\{ \int X\dot{x} e^{\int A\dot{x}} + c \right\};$$

which is a general formula applicable whatever be the functions A and X .

$$\text{Now by the question } A = \frac{3}{10 - 5x}, X = \frac{2x}{10 - 5x};$$

therefore $-\int A\dot{x} = \frac{1}{5} \int \frac{-\dot{x}}{2 - x} = \frac{1}{5} \log(2 - x)$. Hence

$$e^{-\int A\dot{x}} = (2 - x)^{\frac{1}{5}} \text{ and } e^{\int A\dot{x}} = \frac{1}{(2 - x)^{\frac{1}{5}}}; \text{ and } \int X\dot{x} e^{\int A\dot{x}}$$

$$= \int \frac{2x\dot{x}}{5(2 - x)^{\frac{1}{5}}} = \frac{4}{3(2 - x)^{\frac{3}{5}}} + (2 - x)^{\frac{1}{5}}, \text{ and}$$

$$y = e^{-\int A\dot{x}} \left\{ \int X\dot{x} e^{\int A\dot{x}} + c \right\} = \frac{4}{3} + 2 - x + c(2 - x)^{\frac{1}{5}}.$$

Hence the equation of the fluents is

$$(y + x - \frac{4}{3})^5 - c(2 - x)^5 = 0$$

and here c is put instead of c^3 , which makes no difference, as c is an arbitrary quantity.

THIRD SOLUTION, by Mr. CUNLIFFE.

Assume $y = A + Bx + v$ where A, B are constant quantities, and v is a variable quantity to be determined.

Then the given equation will become

$$2x\dot{x} - 3A\dot{x} - 3Bx\dot{x} - 3v\dot{x} = 10B\dot{x} + 10\dot{v} - 5Bx\dot{x} - 5v\dot{v}.$$

By equating the like terms $-3A = 10B$, or $A = -\frac{10B}{3}$;
 $2 = -2B$, or $B = -1$; and hence $A = -\frac{10B}{3} = \frac{10}{3}$.

Again, from the equality of the remaining terms of the expression, viz. $-3v\dot{x} = 10\dot{v} - 5v\dot{v}$ we get

$$\frac{\dot{v}}{v} = \frac{2}{3} \times \frac{-\dot{x}}{2-x};$$

and taking the fluents

$$\text{hyp. log. } v = \text{hyp. log. } \left\{ d \times (2-x) \right\}^{\frac{2}{3}}, \text{ or } v = \left\{ d \times (2-x) \right\}^{\frac{3}{2}}.$$

$$\text{Whence } y = A + Bx + v = \frac{10}{3} - x + \left\{ d \times (2-x) \right\}^{\frac{3}{2}},$$

where d is a necessary correction to be determined from the circumstances of the problem.

FOURTH SOLUTION, by HYPATIA.

From the proposed equation we have

$$(2x - 3y)\dot{x} - (10 - 5x)\dot{y} = 0.$$

Put $x = 2 + v$, $y = \frac{2}{3} - z$, then $\dot{x} = \dot{v}$ and $\dot{y} = -\dot{z}$, and the equation becomes

$$(2v + 3z)\dot{v} - 5v\dot{z} = 0,$$

which admits of being expressed thus,

$$5v(\dot{v} - \dot{z}) - 3(v - z)\dot{v} = 0.$$

$$\text{Hence } \frac{5(\dot{v} - \dot{z})}{v - z} - \frac{3\dot{v}}{v} = 0;$$

therefore $5 \log. (v - z) - 3 \log. v = \log. c$ (c a constant given quantity), and $(v - z)^5 - cv^3 = 0$,

$$\text{therefore } (x - \frac{1}{3} + y)^5 - c(x - 2)^3 = 0$$

the equation required.

FIFTH

FIFTH SOLUTION, *by* SCOTICUS.

Let the proposed equation be written thus :

$$\dot{y} + \frac{3y}{10-5x} \dot{x} = \frac{2x\dot{x}}{10-5x}.$$

Now it is certain from the principles of the fluxional calculus that there is some function of x by which if we multiply the expression $\dot{y} + \frac{3y}{10-5x} \dot{x}$ the product shall be a complete fluxion ;

let this function be X ; therefore $X\dot{y} + \frac{3Xy}{10-5x} \dot{x}$ must be a

complete fluxion. But is known, that when any expression of this form $M\dot{x} + N\dot{y}$ (where M and N denote any functions of x and y) is a complete fluxion, then the fluxion of M taken upon the supposition that y only is variable, and divided by $\dot{y}$ is equal to the fluxion of N , taken upon the supposition that x only

is variable, and divided by $\dot{x}$. In the present case $M = \frac{3Xy}{10-5x}$,

$N = X$, hence $\frac{\dot{M}}{\dot{y}} = \frac{3X}{10-5x}$, $\frac{\dot{N}}{\dot{x}} = \frac{\dot{X}}{\dot{x}}$, therefore $\frac{3X}{10-5x}$

$= \frac{\dot{X}}{\dot{x}}$, and $\frac{\dot{X}}{X} = \frac{1}{2} \cdot \frac{\dot{x}}{2-x}$, and $X = (2-x)^{-\frac{1}{2}}$, therefore

the proposed equation when prepared for integration is

$$(2-x)^{-\frac{1}{2}} \dot{y} + \frac{1}{2} (2-x)^{-\frac{3}{2}} y \dot{x} = \frac{1}{2} x \dot{x} (2-x)^{-\frac{5}{2}};$$

and the equation of the fluents is

$$(2-x)^{-\frac{1}{2}} y = \frac{1}{2} (2-x)^{-\frac{1}{2}} + (2-x)^{\frac{5}{2}} + c,$$

which by proper reduction becomes

$$\frac{(3y + 3x - 10)^{\frac{1}{2}}}{(2-x)^{\frac{3}{2}}} = \text{a constant quantity.}$$

SIXTH SOLUTION, *by* Mr. IVORY.

Let the equation be brought to this form

$$\dot{y} + y \times \frac{1}{2} \cdot \frac{\dot{x}}{2-x} = \frac{1}{2} \cdot \frac{x\dot{x}}{2-x} :$$

L 2

assume

assume $\frac{\dot{v}}{v} = \frac{1}{2} \cdot \frac{\dot{x}}{2-x}$: And by substitution we get

$$\dot{y} + y \times \frac{\dot{v}}{v} = \frac{1}{2} \cdot \frac{\dot{x}}{2-x}: \text{ And by multiplying by } v,$$

$$v\dot{y} + y\dot{v} = \frac{1}{2} \cdot \frac{x\dot{v}}{2-x} \times v: \text{ And by integrating,}$$

$$v \times y = C + \int \frac{1}{2} \cdot \frac{x\dot{v}}{2-x} \times v.$$

Because $\frac{\dot{v}}{v} = \frac{1}{2} \cdot \frac{\dot{x}}{2-x}$, therefore $v = (2-x)^{-\frac{1}{2}}$:

$$\text{Consequently } \int \frac{1}{2} \cdot \frac{x\dot{x}}{2-x} \times v = \int \frac{1}{2} \cdot \frac{x\dot{x}}{2-x} \times (2-x)^{-\frac{1}{2}} = \\ \int \frac{1}{2} \cdot x\dot{x} \times (2-x)^{-\frac{3}{2}} = (\text{by substituting } 2 - (2-x) \text{ for } x \\ \int \frac{1}{2} \dot{x} (2-x)^{-\frac{3}{2}} - \int \frac{1}{2} \dot{x} (2-x)^{-\frac{3}{2}} = \frac{1}{2} (2-x)^{-\frac{1}{2}} + (2-x)^{\frac{1}{2}}.$$

Therefore by substitution,

$$(2-x)^{-\frac{1}{2}} \times y = C + \frac{1}{2} (2-x)^{-\frac{1}{2}} + (2-x)^{\frac{1}{2}}:$$

And, by multiplying by $(2-x)^{\frac{1}{2}}$, we get

$$y = C \times (2-x)^{\frac{1}{2}} + \frac{1}{2} + 2 - x,$$

$$\text{or } y = \frac{1}{2} - x + C \times (2-x)^{\frac{1}{2}}.$$

N. B. If P and Q denote any functions of x , the equation $\dot{y} + P\dot{x} \times y = Q\dot{x}$ may be integrated by the substitution we have made use of. For, by putting $\frac{\dot{v}}{v} = P\dot{x}$, we readily get $v \times y = C + \int Q\dot{x} \times v$; where the fluent $\int Q\dot{x} \times v$ depends only on the variable quantity x .

Solutions were also received from Messrs. Cavill and Toplis.

IX. QUESTION 159, by A. M.

If from the extremities A and B of the diameter AB of a given semicircle ABC, two chords, AC, BD, be drawn intersecting each other in P, and BD bisecting the arc AC in D; it is required to determine the equation and quadrature of the curve which is the locus of the point P?

SOLUTION,

SOLUTION, by Mr. JOHN WHITLEY.

Fig. 57. pl. 3. Let the figure be constructed as per question : Join BC and draw PM perpendicular to AB. Since the arc AC is bisected in D, the angle ABD is equal to the angle DBC ; and because the angles BMP, BCP are right angles, therefore the triangles PBM, PBC are similar and in all respects equal. Hence PM = PC and BM = BC. By similar triangles AB : BM = BC :: AP : PM. Hence if AB be put = d , BM = x and PM = y , we have

$$d : x :: AP = \sqrt{\{(d-x)^2 + y^2\}} : y$$

$$\text{or, } d^2 : x^2 :: (d-x)^2 + y^2 : y^2;$$

therefore $\pm y = x \sqrt{\left(\frac{d-x}{d+x}\right)}$ the equation of the curve.

When $\pm y = 0$, we have $d-x = 0$, or $x = d$ which shews that the curve passes through A. When $x = 0$, $\pm y = 0$; therefore B is a *punctum duplex*.

If λ be negative, or M be in AB produced to the left of B, then $\pm y = x \sqrt{\left(\frac{d+x}{d-x}\right)}$. And hence it appears that

when $x = d$, the value of y becomes infinite, and consequently if BK be taken = AB, and XX be drawn perpendicular to AK at K, then XX will be an asymptote to the curve.

$$\text{The fluxion of the area is } y\dot{x} = \frac{dx\dot{x} + x^2\dot{x}}{\sqrt{(d^2 - x^2)}},$$

the correct fluent of which is

$$d^2 - d \sqrt{(d^2 - x^2)} \mp \frac{1}{2} dv \pm \frac{x}{2} \sqrt{(d^2 - x^2)},$$

where v is a circular arc whose radius is d and sine x ; and when $x = d$, if $n = .7854$, &c. we shall have the whole area of the *nodus*, or that part of the curve within the circle equal to $2d^2 (1 - n)$; and the area of the part without the circle, or the space included by the curve P B r' and the asymptote XX, equal to $2d^2 (1 + n)$. Hence the whole area of the curve or the required quadrature is equal to $4d^2$, that is, equal to four times the square of the diameter AB of the given circle.

This curve is of the same species as those in questions 142 and 157, and may be constructed in various ways; but the following construction is perhaps as simple as any: Let an indefinite tangent QA be drawn to the circle at the point A, and
from

from B, let any straight line BDQ be drawn meeting the circle in D and the tangent in Q; from D, along DB, set off DP equal to QD, and D will be a point in the curve.

Neat Solutions were also received from Messrs. Cavill, Nicholson, Swale and Toplis.

X. QUESTION 160, by Mr. CUNLIFFE, R. M. College.

Required the sum of the infinite series

$$1 - \frac{1}{2} \times \left(\frac{1}{3}\right) + \frac{1}{3} \times \left(\frac{1}{3} + \frac{1}{3}\right) - \frac{1}{4} \times \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + \frac{1}{5} \times \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) - \&c.$$

FIRST SOLUTION, by Mr. WALLACE.

$$\text{Let } x^2 - \frac{1}{2} \cdot \frac{x^4}{2} + \left(\frac{1}{3} + \frac{1}{3}\right) \frac{x^6}{3} - \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) \frac{x^8}{4} + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) \frac{x^{10}}{5}$$

— &c. be the expansion of y , a function of a variable quantity x : the sum of the proposed series is evidently the value of y corresponding to $x = 1$; therefore, to find that sum we must investigate the function from the above expression for its expansion.

From the equation

$$y = x^2 - \frac{1}{2} \cdot \frac{x^4}{2} + \left(\frac{1}{3} + \frac{1}{3}\right) \frac{x^6}{3} - \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) \frac{x^8}{4} + \&c.$$

by taking the fluxions and dividing by $2x$ we have

$$\frac{\dot{y}}{2x} = x - \frac{1}{2}x^3 + \left(\frac{1}{3} + \frac{1}{3}\right)x^5 - \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)x^7 + \&c.$$

Let the first term of the series be transposed to the other side of the equation, and let the whole be multiplied by x^2 , thus we have

$$\frac{\dot{y}x^2}{2x} - x^3 = -\frac{1}{2}x^5 + \left(\frac{1}{3} + \frac{1}{3}\right)x^7 - \&c.$$

Let each side of this last equation be added to the corresponding side of the preceding equation and the result is

$$\frac{\dot{y}}{2x} (1 + x^2) - x^3 = x - \frac{1}{2}x^3 + \frac{1}{3}x^5 - \frac{1}{4}x^7 + \&c.$$

Therefore

$$\dot{y} = \frac{2x^3}{1+x^2} + \frac{2x}{1+x^2} \left\{ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c. \right\}$$

Now if ϕ denote the arch whose tangent is x , (radius being unity)

unity) then it is known that $\varphi = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c.$

and $\dot{\varphi} = \frac{\dot{x}}{1+x^2}$, therefore

$$\dot{y} = \frac{2\dot{x}x^2}{1+x^2} + 2\dot{\varphi}\varphi = 2x\dot{x} - \frac{2x\dot{x}}{1+x^2} + 2\dot{\varphi}\varphi$$

and taking the fluents, so that $y = 0$ when $x = 0$, $y = x^2 - \text{hyp. log. } (1+x^2) + \varphi^2$.

But when $x = 1$, then $\varphi = \frac{1}{2}\pi$; (here $\pi = 3.14159$) therefore the sum of the proposed series is $1 - \text{h. log. } 2 + \frac{1}{16}\pi^2$.

Scholium. In the same manner we may find the sum of the series

$1 - \left(\frac{1}{m}\right)\frac{1}{n} + \left(\frac{1}{m} + \frac{1}{m+i}\right)\frac{1}{n+d} - \left(\frac{1}{m} + \frac{1}{m+i} + \frac{1}{m+i^2}\right)\frac{1}{m+2d} + \&c.$ which may be considered as the value of this other series $1 - \left(\frac{1}{m}\right)\frac{x^{in}}{n} + \left(\frac{1}{m} + \frac{1}{m+i}\right)\frac{x^{i(n+d)}}{n+d} - \left(\frac{1}{m} + \frac{1}{m+i} + \frac{1}{m+i^2}\right)\frac{x^{i(n+2d)}}{n+2d} + \&c.$ when x is supposed $= 1$. Let y denote the function which being expanded produces this last series, then, proceeding in all respects as before, we find

$$y = \frac{i\dot{x}}{1+x^{id}} \left\{ -\frac{x^{in-1}}{m} + \frac{x^{i(n+d)-1}}{m+i} - \frac{x^{i(n+2d)-1}}{m+2i} + \&c. \right\}$$

But the series in the parenthesis is the expansion of

$$-dx^{in-dm-1} \times \int \frac{\dot{x}x^{md-1}}{1+x^{di}},$$

Let this expression be denoted by X , then

$$y = \int \frac{i\dot{x}X}{1+x^{id}}$$

the fluent being so taken that $\dot{x} = a$ when $x = 0$.

SECOND SOLUTION, by Mr. CUNLIFFE, the Proposer.

Assume $\int \frac{\dot{x}}{1+x^2} \times (x - ax^3 + bx^5 - cx^7 + dx^9 - ex^{11} + \&c.)$
 $= \frac{x^2}{2} - \frac{1}{4}Px^4 + \frac{1}{4}Qx^6 - \frac{1}{4}Rx^8 + \frac{1}{16}Sx^{10} - \frac{1}{16}Tx^{12} \&c.$

taking

taking the fluxions and dividing by $\frac{\dot{x}}{1+x^2}$

$$\begin{aligned} x-ax^3+bx^5-cx^7+dx^9\&c. &= (1+x^2) \times (x-Px^3+Qx^5-Rx^7\&c.) \\ &= x - Px^3 + Qx^5 - Rx^7 + Sx^9 - Tx^{11} \&c. \\ &+ x^3 - Px^5 + Qx^7 - Rx^9 + Sx^{11} \&c. \end{aligned}$$

making the coefficients of the like terms equal to each other,
 $P-1=a$ or $P=1+a$; $Q-P=b$, or $Q=P+b=1+a+b$; $R-Q=c$, or $R=Q+c=1+a+b+c$ &c. therefore

$$\begin{aligned} \int \frac{\dot{x}}{1+x^2} \times (x - ax^3 + bx^5 - cx^7 + dx^9 - ex^{11} \&c.) &= \\ \frac{x^2}{2} - \frac{1}{4}x^4(1+a) + \frac{1}{6}x^6(1+a+b) - \frac{1}{8}x^8(1+a+b+c) + \frac{1}{10}x^{10} & \\ (1+a+b+c+d) \&c. \end{aligned}$$

Take $a=\frac{1}{3}$, $b=\frac{1}{5}$, $c=\frac{1}{7}$, &c. then $x-ax^3+bx^5-cx^7\&c.$
 $= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \&c. = \int \frac{\dot{x}}{1+x^2} = z$, where z
denotes the length of a circular arc, radius 1 and tangent x .

$$\begin{aligned} \text{Therefore } \int \frac{\dot{x}}{1+x^2} \times (x - ax^3 + bx^5 - cx^7 + dx^9 - ex^{11} \&c.) &= \\ = \frac{z^2}{2} = \frac{x^2}{2} - \frac{1}{4}Px^4 + \frac{1}{6}Qx^6 - \frac{1}{8}Rx^8 + \frac{1}{10}Sx^{10} - \frac{1}{12}Tx^{12} \&c. & \\ \text{or } z^2 = x^2 - \frac{1}{4}Px^4 + \frac{1}{6}Qx^6 - \frac{1}{8}Rx^8 + \frac{1}{10}Sx^{10} - \frac{1}{12}Tx^{12} \&c. & \\ = x^2 - \frac{1}{4}x^4(1+\frac{1}{3}) + \frac{1}{6}x^6(1+\frac{1}{3}+\frac{1}{5}) - \frac{1}{8}x^8(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}) \&c. & \end{aligned}$$

Take $x=1$, then $\frac{q^2}{4} = 1 - \frac{1}{4}(1+\frac{1}{3}) + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5}) - \frac{1}{8}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7})$
 $+ \frac{1}{10}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}) - \frac{1}{12}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}) \&c.$
where q denotes the length of the quadrantal arc of a circle
radius 1.

And therefore $1 + \frac{q^2}{4} = 2 - \frac{1}{4}(1+\frac{1}{3}) + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5}) - \frac{1}{8}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7})$
&c.

$$\text{also h. l. (2) } = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \&c.$$

and taking the difference

$$1 + \frac{q^2}{4} - \text{h. l. (2)} = 1 - \frac{1}{2}(\frac{1}{3}) + \frac{1}{3}(\frac{1}{3}+\frac{1}{5}) - \frac{1}{4}(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}) \&c.$$

ad infinitum.

THIRD SOLUTION, *by Mr. IVORY.*

Put $S = z^2 - \frac{1}{2} \cdot \frac{z^4}{2} + (\frac{1}{2} + \frac{1}{2}) \frac{z^6}{3} - (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \frac{z^8}{4} + \&c.$

And, by taking the fluxions,

$$\frac{1}{2} \times \frac{dS}{dz} = z - \frac{1}{2} \cdot z^3 + (\frac{1}{2} + \frac{1}{2}) z^5 - (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) z^7 + \&c.$$

And, by multiplying by $1 + z^2$,

$$\frac{1}{2} \times \frac{dS}{dz} \times (1 + z^2) = z^3 + z - \frac{1}{2} z^3 + \frac{1}{2} z^5 - \frac{1}{2} z^7 + \&c.$$

Let $z = \text{tangent } \varphi$, then $\varphi = z - \frac{1}{2} z^3 + \frac{1}{2} z^5 - \frac{1}{2} z^7 + \&c.$

And, by substitution, $\frac{1}{2} \times \frac{dS}{dz} \times (1 + z^2) = z^3 + \varphi$:

And, by multiplying by $\frac{2dz}{1 + z^2} = 2d\varphi$,

$$dS = \frac{2z^3 dz}{1 + z^2} + 2\varphi d\varphi:$$

And by integrating $S = z^2 - \log. (1 + z^2) + \varphi^2$, no constant quantity being necessary. Let z be taken $= 1$, then the

sum of the series proposed to be summed $= 1 - \log. 2 + \frac{\pi^2}{16}$,

where $\pi = 3.14159 \&c.$

Mr. Cavill also answered it.

XI. QUESTION 161, *by Mr. W. SIMPSON, Bolton.*

Let ACB be a given triangle, D, L, Q, given points in the side AB; through D draw any right line DEF, cutting AC, BC, in E and F respectively, and through E and F draw right lines LEP, QFP intersecting in P. Required the locus of P?

FIRST SOLUTION, *by Mr. WALLACE.*

Fig. 58, pl. 3. Through C draw RCH parallel to AB; produce DE, QF, LE to meet RH in H, K, and R, and join CP meeting AB in G.

The triangles HFC, DFB, have their bases HC, BD, similarly divided at K and Q, therefore $HC : KC :: DB : BQ$; but DB and BQ being given lines, the ratio of DB to BQ is given; therefore the ratio of HC to KC is given. Again, the triangles HER, DEL, have their bases HR, DL, similarly divided at C and A, therefore $HC : CR :: DA : AL$; now the ratio of DA to AL is given, for DA and AL are given, therefore the ratio of HC to CR is given; but it has been shewn that the ratio of HC to CK is given; therefore the ratio of CK to CR is given. Now the triangles QPL, RPK have their bases QL, RK, similarly divided at G and C, therefore $QG : GL :: KC : CR$; but the ratio of KC to CR has been shewn to be given, therefore the ratio of QG to GL is given; and Q and L being by hypothesis given points, the point G is given, and the line CG is given by position. Therefore the locus of the point P is a straight line CG given by position.

SECOND SOLUTION, by Mr. LOWRY, R. M. College.

Fig. 59, pl. 3. Draw CP to meet AB in K, and through P draw GO parallel to AB, meeting DE in G, AC in H, and BC in O.

Then by similar triangles $HP : GP :: AL : DL$,
and $GP : PO :: DQ : BQ$;
therefore by compounding $HP : PO :: AL \cdot DQ : DL \cdot BQ$,
which is a given ratio, because AL, DQ, DL, and BQ are given lines; and since HO is parallel to the base AB, and is divided at P in a given ratio, the locus of the point C is a straight line passing through C, and dividing the base AB into two parts AK, BK, which have the same given ratio that HP has to PO.

The same algebraically.

Put $DL = a$, $QB = b$, $DQ = c$, $AL = d$, $AB = m$, $AC = n$, $HC = x$, and $PH = y$; then by similar triangles

$$AL = d : PH = y :: LD = a : PG = \frac{ay}{d},$$

$$\text{and } DQ = c : PG = \frac{ay}{d} :: QB = b : PO = \frac{aby}{cd};$$

$$\text{Hence } HO = y + \frac{aby}{cd} = \frac{cd + ab}{cd} \times y.$$

$$\text{And } AB = m : AC = n :: HO : HC = \frac{cd + ab}{mcd} \times ny,$$

or $x = \frac{cd + ab}{mcd} \times ny$, an equation to a straight line.

When

When $x = 0$, y is $= 0$, and the locus passes through C ; also when $x = n$, y is $= \frac{mcd}{cd + ab}$; hence if AK be taken $=$ to $\frac{mcd}{cd + ab}$, the line CK will be the locus required.

XII. QUESTION 162, by AMICUS.

If a given weight be sustained by means of a string whose ends are connected (by loops or rings) to two rods placed in the same vertical plane, and making given angles with the horizon; it is required to ascertain the position of the said weight when it rests in equilibrio, supposing it has liberty to slide freely along the string ?

SOLUTION, by AMICUS, the Proposer.

Fig. 60, pl. 3. Let P and Q be the points where the rings press upon the rods CA, CB ; and W the position of the weight when it is at rest. Draw AB parallel to the horizon, and WG perpendicular to AB. Then since the rings and weights are to be stationary, the strings PW, QW must be perpendicular to the rods; for if they were in any other directions as Wp, Wq, then the forces acting in those oblique directions might be each decomposed into two others, pP, PW, and Qq, QW ; and as there is no force to counteract the effect of either of the forces, pP, Qq, which act in the direction of the rods, the rings would have a tendency to move in those directions till they came to the points P, Q, where the strings are perpendicular to the rods. Also, since the weight W is at liberty to slide freely along the string, the forces acting in the direction of the strings WP, WQ must be equal; for if the one force was greater than the other, the weight would have a tendency to move towards that side to which it was drawn by the greater force.

Now when the forces in the directions PW, QW are equal, their resultant (which is directly opposed to the action of gravity on the weight W) will bisect the angle PWQ, or the angle PWG will be equal to the angle QWG ; and, therefore, since GW is perpendicular to AB, the angle AWP will be equal to the angle BWP, which (because the angles at P and Q are right angles) can only be the case when the angle WAP is equal to the angle WBQ, or when the rods are equally inclined to the horizon.

In every other case the weight will have a tendency to slip along the string till it comes to one of the rings, and then the string will be perpendicular to the other rod.

To find the position of the weight, when the rods are equally inclined to the horizon, draw AD perpendicular to CB, meeting WE, a parallel to CB, in E; then since the angles DAB, QWB, AWP are equal, the right-angled triangles APW, AEW are equal in all respects; and therefore AE is = to PW; and ED is = to WQ by reason of the parallels; therefore AD is = to PW + WQ, or the length of the string. Hence it is evident that if AD be drawn perpendicular to CB, so as to be equal to the length of the string, and AB be drawn parallel to the horizon, or CB be taken equal to CA; then the point W may be any where in the line AB; for if the lines WP, WQ, be perpendicular to the rods they will make equal angles with AB, and their sum will always be equal to the length of the string.

The pressure of the rings on each rod is equal to the weight W multiplied by the sine of the angle CAB.

A similar result may be obtained by finding the locus of the point W, without taking into consideration the equality of the forces acting in the direction of the strings. For since the sum of the perpendiculars WP, WQ, is equal to a given line, the locus of the point W is a straight line AB given by position, as is pretty generally known; and as the weight can only be at rest when it has descended as low as it possibly can; it is evident that when the locus AB is parallel to the horizon, the weight will rest at any assigned point of the line AB; but if it be inclined to the horizon, then the weight will continue to descend till it comes to the lowest point of the locus which will be at one of the rings.

If one end of the thread was fastened at any point Q in the rod BC, while the other end was at liberty to slip along AC, the string PW would still be perpendicular to CA, and the line WQ would bisect the angle FWQ, as before; and therefore the angle BWQ would be equal to the angle PWA, or to the complement of the given angle WAP, and since the angle WBQ is given, the angle WQB would also be given: and the position of the point W might be determined by drawing QW to make the given angle with BC, and taking QI = to the length of the string; then PI, drawn perpendicular to the horizon, will meet AC in the point P; and PW, drawn perpendicular to AC, will meet IQ in W, the point required. For it is evident that the triangle IWP is isosceles, and therefore $PW + WQ = IQ =$ to the length of the string.

XIII. QUESTION 163, by Mr. CUNLIFFE.

Find such positive values of x , y , and z , in whole numbers, as will make the formulæ

$x^2 + xy + y^2$, $x^2 + xz + z^2$ and $y^2 + yz + z^2$,
all rational squares?

SOLUTION, by Mr. CUNLIFFE, the Proposer.

Assume $x + y = a$ for the root of the first, and $x + z = b$ for the root of the second;

then $x^2 + xy + y^2 = (x + y - a)^2 = x^2 + 2xy + y^2 - 2a(x + y) + a^2$,

and $x^2 + xz + z^2 = (x + z - b)^2 = x^2 + 2xz + z^2 - 2b(x + z) + b^2$;

from the first of which $x = \frac{2ay - a^2}{y - 2a}$;

and from the second $x = \frac{2bz - b^2}{z - 2b}$;

therefore $\frac{2ay - a^2}{y - 2a} = \frac{2bz - b^2}{z - 2b}$; whence $y = \frac{z(a^2 - 4ab) - 2ab(a - b)}{2z(a - b) - 4ab + b^2}$;

$y + z = \frac{z(a^2 - 4ab) - 2ab(a - b)}{2z(a - b) - 4ab + b^2} + z$

$= \frac{2z^2(a - b) + z(a^2 - 8ab + b^2) - 2ab(a - b)}{2z(a - b) - 4ab + b^2}$, and by

means of what has been deduced $y^2 + yz + z^2 = (y + z)^2 - yz$

$= \frac{\{2z^2(a - b) + z(a^2 - 8ab + b^2) - 2ab(a - b)\}^2}{\{2z(a - b) - 4ab + b^2\}^2}$

$= \frac{\{z^2(a^2 - 4ab) - 2abz(a - b)\} \times \{2z(a - b) - 4ab + b^2\}}{\{2z(a - b) - 4ab + b^2\}^2} = a$

square; therefore $\{2z^2(a - b) + z(a^2 - 8ab + b^2) - 2ab(a - b)\}^2 =$

$\{z^2(a^2 - 4ab) - 2abz(a - b)\} \times \{2z(a - b) - 4ab + b^2\} = 4z^4(a - b)^2$

$+ 2z^3(a - b)(a^2 - 12ab + 2b^2) + z^2(a^4 - 16a^3b + 57a^2b^2 -$

$16ab^3 + b^4) - 2abz(a - b)(2a^3 - 12ab + b^2) + 4a^2b^2(a - b)^2$

$= a$ square.

Assume

Assume $2z^2(a-b) + z\left(\frac{a^2-12ab+2b^2}{2}\right) - 2ab(a-b)$ for its root :

Then $4z^4(a-b)^2 + 2z^3(a-b)(a^2-12ab+2b^2) + z^2(a^4-16a^3b+57a^2b^2-16ab^3+b^4) - 2abz(a-b)(2a^2-12ab+b^2) + 4a^2b^2(a-b)^2 =$
 $\left\{2z^2(a-b) + z\left(\frac{a^2-12ab+2b^2}{2}\right) - 2ab(a-b)\right\}^2 = 4z^4(a-b)^2$
 $+ 2z^3(a-b)(a^2-12ab+2b^2) + \frac{z^2(a^2-12ab+2b^2)^2}{4} - z^2 \times 8ab$
 $(a-b)^2 - 2abz(a-b)(a^2-12ab+2b^2) + 4a^2b^2(a-b)^2,$
 which after proper reduction gives $\frac{8b(a-b)(a^2-b^2)}{3a^4-8a^2b+16ab^2+16b^3},$
 where a and b may be taken at pleasure.

Example 1. Take $a = 3$, and $b = 2$, then $z = \frac{80}{257}$,
 $x = \frac{b^2 - 2bz}{2b - z} = \frac{59}{79}$, and $y = \frac{a^2 - 2ax}{2a - x} = \frac{357}{415}$; which
 fractions being reduced to a common denominator, and that
 denominator rejected will give $z = 80 \times 79 \times 415 = 2622800$;
 $y = 357 \times 257 \times 79 = 7248171$ and $x = 59 \times 257 \times 415 =$
 6292645 , which are three numbers that will answer.

Example 2. Take $a = 4$, and $b = 3$, then $z = \frac{7}{34}$, $x = \frac{264}{197}$,
 and $y = \frac{65}{72}$, and ordering these fractions as in the first
 example we may take $z = 7 \times 197 \times 72$, $x = 264 \times 34 \times 72$,
 and $y = 65 \times 197 \times 34$, or the halves of these numbers will
 do, viz. $x = 264 \times 34 \times 36 = 323136$, $y = 65 \times 197 \times 17$
 $= 217685$, and $z = 7 \times 197 \times 36 = 49644$, which numbers
 will answer, and are something smaller than the former set.

Again, assume $2z^2(a-b) + \frac{z}{2}(a^2-12ab+2b^2) + 2ab$
 $(a-b)$ for the root of the general expression which is to be
 made a square, that is, put $4z^4(a-b)^2 + 2z^3(a-b)(a^2-12ab+2b^2) + z^2(a^4-16a^3b+57a^2b^2-16ab^3+b^4)$
 $- 2abz(a-b)(2a^2-12ab+b^2) + 4a^2b^2(a-b)^2 =$
 $\left\{2z^2(a-b) + \frac{z(a^2-12ab+2b^2)}{2} + 2ab(a-b)\right\}^2 = 4z^4$
 $(a-b)^2 + 2z^3(a-b)(a^2-12ab+2b^2) + \frac{z^2(a^2-12ab+2b^2)^2}{4}$
 $+ z^2$

+ $z^3 \times 8ab (a - b)^2 + 2abz (a - b) (a^2 - 12ab + 8b^2) + 4a^2b^2 (a - b)^2$, which after proper reduction gives

$z = \frac{8b(a-b)(a^2 - 8ab + b^2)}{a^3 - 24a^2b + 48ab^2 - 10b^3}$ which is a second formula for the value of z .

Example 1. Take $b = 2$ and $a = 1$, then $z = \frac{176}{17}$, $x = \frac{53}{9}$, and $y = \frac{97}{35}$; and ordering these fractions as in the former examples we may take $z = 176 \times 9 \times 35 = 55440$, $x = 53 \times 17 \times 35 = 31535$, and $y = 97 \times 9 \times 17 = 14841$; and these numbers are still less than either of the preceding sets.

Example 2. Take $b = 7$ and $a = 4$; then $z = \frac{7 \times 53}{18}$, $x = \frac{616}{17}$ and $y = \frac{97}{10}$; and proceeding with these fractions as before we shall have, after dividing the numbers by their greatest common measure, $z = 371 \times 17 \times 5 = 31535$, $x = 616 \times 9 \times 10 = 55440$, and $y = 97 \times 17 \times 9 = 14841$ which are the very numbers found in the preceding example of the present formula.

XIV. QUESTION 164, by URSA MINOR.

A straight line drawn through the focus of any conic section, to the point in which a diameter meets the directrix, will be perpendicular to a tangent at either extremity of that diameter?

FIRST SOLUTION, by AMICUS.

Fig. 61, pl. 3. Let the diameter DC meet the directrix AQ in A, and draw AF to the focus F; then AF will be perpendicular to GH, a tangent at either extremity of the diameter.

When the section is an ellipse, or a hyperbola, let f be the other focus; join fC , FC , and DF , and draw CP , DQ perpendicular to the directrix. Let the tangent GC meet AF in H, and DF produced in E; then the triangles DFB, fCB , having two sides of the one equal to two sides of the other, and the angles at B equal, they are equal in all respects: therefore the angle fCD is equal to the angle CDF; and fC is parallel to DE; therefore

therefore the angle DEC is equal to the angle GCf, or to the angle FCE, which is equal to the angle GCf, as is proved by every writer on conic sections.

Again, by a general property of the sections $DQ : CP :: FD : CF :: AB : AC$, therefore (Eu. vi. prop. A.) AF bisects the angle CFE, and the triangles CFH, EFH having the angle HCF equal to the angle HEF, and the angle CFH to the angle EFH, have also the angle CHF equal to the angle EHF; and therefore AF is perpendicular to the tangent GH. It is obvious that AF is also perpendicular to the tangent at the other extremity of the diameter: for since tangents at the extremities of any diameter of an ellipse or hyperbola are parallel; a line which is perpendicular to one of the tangents will also be perpendicular to the other.

When the curve is a parabola. Fig. 62, pl. 3. Let AT meet the tangent in H and join CF, then because FC is equal to CA and the angle TCH to ACH, the triangles FCH, ACH are equal in every respect, therefore the angle AHC is equal to the angle FHC, and AF is perpendicular to the tangent CH.

SECOND SOLUTION, by Mr. JOHN WHITLEY.

Fig. 63, pl. 3. Let GH, a straight line drawn through the focus F, of an ellipse PHL, meet the directrix GT in G, the point in which the diameter PL meets it; the straight line GH, shall be perpendicular to a tangent at either extremity of the diameter PL. Draw PS to the other focus S, and through F, the straight line LFI. Then because SC is equal to FC, and PC to LC, therefore PS is equal and parallel to FL; wherefore the angle IFP is equal to the angle SPF. Draw PO perpendicular to a tangent to the curve at P or L: then PO will bisect the angle SPF (cor. to prop. x. B. I. Emerson's Conics). And (by prop. LXIII. *ibid.*) the straight line GF also bisects the angle IFP; therefore the angle GFP is equal to the angle OPF; and consequently GF is parallel to PO, or perpendicular to a tangent at either extremity of the diameter PL. And the same is manifestly true of any other conic section whatever.

XV. QUESTION 165, by AMICUS.

If one end of a given beam rest against a smooth vertical plane, and the other end be sustained by a weight fastened to a string, which passes over a pulley placed at a given point; it is required to determine the position of the beam when it rests in equilibrium?

FIRST

FIRST SOLUTION, *by Mr. LOWRY.*

Let AB (fig. 64, pl. 3.) represent the beam, C its centre of gravity, EG the vertical plane, P the pulley given in position, W the weight fastened to the end of the string, and w the weight of the beam. Let the vertical line IC meet AP in I and join BI; then the beam is acted upon by three forces, *viz.* the weight W acting at A in the direction AI, the weight w acting at C in the direction IC, and the reaction of the plane against the end of the beam at B, acting perpendicularly to the plane; and as these forces keep the beam in equilibrio they necessarily tend to the same point I, or the reaction of the plane is exerted in the direction IB, and consequently IB is perpendicular to EG.

Draw AD parallel to IB, meeting IC produced in D; then the forces acting at A, C, and B in the directions AI, IC, and IB are proportional to the lines AI, ID and AD; or if AI represent the weight W, then ID will represent the weight w , and AD will represent the pressure at B.

Produce AP to meet the plane at E and draw PQ perpendicular to EG; then, since the pulley and vertical plane are given in position, the line PQ will be given in magnitude, and the point Q will be given in position; and the right-angled triangles QEP, AID being similar, $PE : EQ :: AI : ID :: W : w$,

$$\text{or } PE = EQ \times \frac{W}{w}; \text{ also } PE^2 - EQ^2 = PQ^2, \text{ or } EQ^2 \times \frac{W^2}{w^2} - EQ^2 = PQ^2; \text{ and therefore } EQ = PQ \times \frac{w}{\sqrt{W^2 - w^2}},$$

which is a given quantity; therefore E is a given point, and the line EPA is given by position, or the angle PEQ (AID) is of a given magnitude.

The position of AB may now be determined either by construction, or by calculation:

And first by construction. Take a line ab = to AB the length of the beam, and let ac be = to AC and cb = to CB; on ac describe a segment of a circle to contain the given angle AID or PEQ, and let it meet the semi-circle described on cb in i ; then if ai , bi , ci be drawn, they will evidently be equal to the lines AI, BI, CI, respectively. On QP produced take QR = bi , and draw RI parallel to QG, meeting AP in I; take QB = RI and IA = ai , and the points A and B are determined. This construction is too obvious to need any particular demonstration in this place.

By calculation. Put $PQ = b$, $EQ = a = b \times \frac{w}{\sqrt{(W^2 - w^2)}}$,
 $AC = m$, $BC = n$, $AB = l$, and $QB = x$: Then by similar
 triangles, $AB : BE :: AC : IC$,

$$\text{or } l : a + x :: m : \frac{m}{l} \times (a + x) = IC.$$

Also $EQ : PQ :: BE : IB$,

$$\text{or } a : b :: a + x : \frac{b}{a} \times (a + x) = IB.$$

And $IC^2 + IB^2 = CB^2$, or $\left(\frac{m^2}{l^2} + \frac{b^2}{a^2}\right)(a + x)^2 = n^2$;

$$\text{Therefore } (a + x)^2 = \frac{n^2}{\frac{m^2}{l^2} + \frac{b^2}{a^2}} = \frac{a^2 l^2 n^2}{m^2 a^2 + l^2 b^2},$$

$$\text{or } a + x = \frac{aln}{\sqrt{(m^2 a^2 + l^2 b^2)}}, \text{ and } x = \frac{aln}{\sqrt{(m^2 a^2 + l^2 b^2)}} - a =$$

$$w \left\{ \frac{ln}{\sqrt{\{l^2(W^2 - w^2) + m^2 w^2\}}} - \frac{b}{\sqrt{(W^2 - w^2)}} \right\}.$$

From this equation we may determine x when the beam and the weight w are given; or we may find what weight W is necessary to retain the beam in any assigned position.

In what has been done, the plane has been considered as perfectly smooth, but the problem may be resolved with the same ease when the friction at any point of the plane is proportional to the pressure of the beam at that point.

In this case, let AB (fig. 65, pl. 3.) represent the beam, and draw AE , IC and PQ as before. Let BH , drawn perpendicular to the plane, represent the pressure at B , and take BL to BH in the proportion of the friction to the pressure (as f to 1): Then BL will represent the friction, or a force equivalent to it, acting in the direction of the plane and opposing the descent of the end of the beam. Complete the parallelogram $B H F L$; then BF will be the resultant of the forces BH , BL ; and when the beam is in equilibrio the line BF must pass through the point I . Draw AD parallel to ID meeting IC produced in D , and PK parallel to IB meeting the plane in K : Then $PQ : QK :: FL : BL :: 1 : f$, or $PK = f \times PQ$ is a given line, and the angle $PKE = IBL = CIB$ is a given angle. Again, the forces acting in the directions AI , IC and IB , are proportional to the lines AI , ID , and AD , or, because of the similar triangles PEK , AID , $PE : EK :: AI : ID :: W : w$, which is a given

ratio; therefore the triangle PKE is given in species, and since PK is given, the line EK and the angle PEK ($\equiv$ AIC) are also given, and the line EA is given in position. Therefore if two circular segments be described on the lines ac , cb , to contain the given angles AIC, CIB and intersect at i , the lines AI, BI will be determined, and the remaining part of the construction will be precisely the same as before.

Calculation. Put $PQ = b$, $EK = a$, $PK = d$, and $x = KB$.

Then, $QK : PQ :: BL : LF :: 1 : f$, or $QK = \frac{b}{f}$ and $PK = \sqrt{b^2 + \frac{b^2}{f^2}} = \frac{b}{f} \sqrt{f^2 + 1} = d$. Also, the sine of the

angle $PKQ = \frac{f}{\sqrt{f^2 + 1}} = s$, and its cosine $= \frac{1}{\sqrt{f^2 + 1}}$

$= c$. Again, $w : W :: a : \frac{W}{w} a = IE$, and by trigonometry,

$PK = d = ac + \frac{a}{w} \sqrt{(W^2 - s^2 w^2)}$, or $a = \frac{dw}{cw + \sqrt{(W^2 - s^2 w^2)}}$.

Now, by similar triangles,

$$l : a + x :: m : \frac{m}{l} \times (a + x) = IC,$$

$$a : d :: a + x : \frac{d}{a} \times (a + x) = IB;$$

and by trigonometry, $IC^2 + IB^2 - 2c \times IC \times IB = CB^2 = n^2$,

$$\text{or } \left(\frac{m^2}{l^2} + \frac{d^2}{a^2} \right) (a + x)^2 - 2c \times \frac{md}{la} (a + x)^2 = n^2;$$

$$\text{therefore } (a + x)^2 = \frac{n^2 l^2 a^2}{a^2 m^2 + d^2 l^2 - 2cmdla},$$

$$\text{and } x = \frac{nla}{\sqrt{(a^2 m^2 + d^2 l^2 - 2cmdla)}} - a.$$

Let AS be drawn perpendicular to ID, then if ID represent the weight of the beam, AS will represent the pressure against the plane, and SD the friction. And, by similar triangles,

$$EK : PQ :: ID : AS = \frac{b}{a} w = \frac{b}{d} \left\{ cw + \sqrt{(W^2 - s^2 w^2)} \right\}$$

$$\text{the pressure against the plane, and } SD = \frac{b}{fd} \left\{ cw + \sqrt{(W^2 - s^2 w^2)} \right\}$$

the friction.

The problem may be resolved in the same manner when the plane makes any given angle with the horizon, only the calculation will be more complicated.

SECOND SOLUTION, *by the Rev. Mr. TOPLIS.*

Let AD (fig. 66, pl. 3.) represent the given beam, DI the vertical plane, B the pulley given in position; suppose W the weight which is connected with A by a string passing over B. From the centre of gravity C of the beam, draw CE representing the weight of the beam; decompose the force CE into the forces CF, FE, one perpendicular and the other parallel to the beam; from D draw DN perpendicular to the beam and equal to $CF \times AC \div AD$; from N draw NK perpendicular to ND, and consequently parallel to DA, and from D draw DK perpendicular to DI and meeting NK in K; then DK will represent the reaction of the plane upon the beam, DN its reaction in a direction perpendicular to the beam, and NK its reaction in a direction parallel to it. From A draw AG perpendicular to the beam and equal to $CF \times CD \div AD$, from G draw GH parallel to the beam and equal $NK + FE$, join AH, then AH will represent the power and direction of the weight W. In this case the system, supposing it in a plane perpendicular to the horizon and vertical plane, is evidently in equilibrio; for, from the property of the lever, the perpendicular forces counteract each other, and the parallel forces NK, FE. are opposed by the force GH which is equal to their sum.

To find the position of the beam by calculation, let us suppose $CE = a$, sine of the angle $CEF = x$, $AC = b$, $CD = c$, $AD = e$.

By trigonometry,

$$\text{Rad. (1) : sin. CEF (x) :: CE (a) : CF} = ax.$$

$$\text{Rad. (1) : cof. CEF } (\sqrt{1-x^2}) :: CE (a) : EF = a\sqrt{1-x^2}.$$

$$\text{But } AG \times AD = CF \times CD, \text{ therefore } AG = \frac{CF \times CD}{AD} =$$

$$\frac{acx}{e}; \text{ likewise } ND \times AD = AC \times CF, \text{ therefore } ND$$

$$= \frac{AC \times CF}{AD} = \frac{abx}{e}. \text{ As MD is parallel to CE and CF}$$

to ND the angle NDM is equal to the angle ECF, consequently the angle NDK is equal to the angle CEF, each of them being the complements of equal angles; therefore as the triangles CEF, NKD have two angles in each equal they are similar. In the triangle

triangle NDK, we have $ND = \frac{abx}{c}$, $\sin. NDK = \sin. CEF = x$, and $\tan. NDK = \frac{\sqrt{(1-x^2)}}{x}$, and, by trigonometry, $\text{rad.}(1) : \tan. KDN \left(\frac{\sqrt{(1-x^2)}}{x} \right) :: ND \left(\frac{abx}{c} \right) : NK = \frac{ab\sqrt{(1-x^2)}}{c}$.

In the triangle AGH, we have $AG = \frac{acx}{c}$, $GH = NK + FE = \frac{ac + ab}{c} \sqrt{(1-x^2)}$, $AH = \sqrt{(AG^2 + GH^2)} = \sqrt{\left\{ \frac{(c+b)^2}{c^2} \times a^2 \times (1-x^2) + \frac{a^2 c^2 x^2}{c^2} \right\}} = W$. If we

suppose $(c+b)^2 \times \frac{a^2}{c^2} = f$, $\frac{a^2 c^2}{c^2} = g$, the expression becomes

$$\sqrt{\{f - (f-g)x^2\}} = W, \text{ or } x = \sqrt{\{(f-W) \div (f-g)\}}.$$

From which equation if W is given, x or the sine of CDI may be found; or if the sine of CDI , the beam's inclination, is given, the weight W may be found.

Let DK be produced to meet BW in P , and draw BM parallel to PD , or perpendicular to the plane; then because the pulley is given in position, the line BM or its equal PD is given, and the angle DSP (equal to the angle CEF) being found, the lines SP , SD may be found, and then AS becomes known.

Also, $\text{rad.} : \sin. AHG = HAD :: AH : AG :: W : \frac{acx}{c}$,
or $\sin. HAD = \frac{acx}{cW} = \frac{ac}{cW} \times \sqrt{\frac{f-W}{f-g}}$, and therefore

the angle HAD is known, and in the triangle ABS the angles and the side AS are known to find BS : then $BS - PS = BP = MD$ the distance of the end of the beam from the point M .

XVI. QUESTION 166, by X. Y.

Suppose two weights connected by a string, which passes over a pulley, and that one of the weights is placed upon an inclined plane, while the other hangs in the air by the connecting string. It is required to determine the *locus* of the centre of gravity of the weights, supposing the relative position of the pulley and plane, the length of the string, and the weights to be all given?

FIRST

FIRST SOLUTION, *by Mr. WALLACE.*

Let a and b be the weights, (fig. 67, pl. 3.) and APB the string connecting them, which passes over the pulley P. And instead of confining the solution of the problem to the case in which one of the weights B is placed upon an inclined plane, let us suppose it placed upon a given line HK of any kind whatever.

Let C be the centre of gravity of the weights. Draw CD parallel to the vertical line PA, meeting PB in D: Then, from similar triangles, AP + PB is to CD + DB as AB to CB, that is, from the nature of the centre of gravity, as $a + b$ to a ; hence the ratio of AP + PB to CD + DB is given; and because AP + PB is given, CD + DB is also given. In the vertical line PA take PE = CD + DB, and then E will be a given point. Draw EF parallel to PB, meeting DC in F, and because DF = PE = DC + DB, therefore CF = BD; now EF = PD, and PD : DB :: AC : CB :: b : a , therefore EF : FC :: b : a ; hence EF has to FC a given ratio. Draw BG parallel to PA, and PG parallel to EC, meeting BG in G; then because DF is equal to DC + DB, so that the point F is below the point C, therefore the point B is also below the point G; and because the triangles PBG, EFC are similar, PB : BG :: EF : FC; but the ratio of EF to FC has been shewn to be given, therefore the ratio of PB to BG is also given. Now P being a given point, and HBK a line given in position, it is manifest that the *locus* of the point G will be a line MGN of a determinate kind, which will depend upon the nature of the given line HBK. Again, because PG is to EC as PB to EF or PD, that is, as AB to AC or as $a + b$ to b , therefore PG has to EC a given ratio; now PG is also parallel to EC, and E is a given point, therefore, whatever be the nature of the line MGN which is the *locus* of the point G, the point C will always be in a line mCn similar to it, and similarly situated; so that the nature of the line MGN being once known, that of mCn the *locus* of the centre of gravity of the weights will also become known.

The solution of the problem thus generalized, which results from the analysis is therefore shortly this: Draw PB to any point B in the line HK; draw BG parallel to PA and in a contrary direction, and take BG such, that PB may have to BG the given ratio of b to a , then determine the nature of MN the *locus* of the point G, and the nature of mn the line which is the *locus* of C, the centre of gravity of the weights will also be determined, for it is similar to MN and similarly situated. And if PL be taken a fourth proportional to $a + b$, a , and AP + PB, then the given points P and E will be similarly situated in respect of

of the similar lines MN , mn ; that is, if from P and E any two parallel straight lines be drawn meeting them in G and C , PG will have to EC a given ratio.

Let us apply this general solution to some particular cases, and first, let us suppose, as in the question, that the weight b is placed at any point B (fig. 68. pl. 3.) upon HBK a straight line given in position. Draw PI perpendicular to HK , and IV parallel to PA , and take IV of such a magnitude that $PI : IV :: b : a :: PB : BG$; then I and V will evidently be given points, and V a point in MN the locus of G ; and because $IV : BG :: PI : PB$, therefore $IV^2 : BG^2 :: PI^2 : PI^2 + IB^2$. Now PI and IV are both given in magnitude, and BG makes a given angle with HK , therefore the point G is manifestly in a hyperbola of which I is the centre, IV a semi-transverse diameter, and HK its conjugate; and it also appears that the semi-conjugate diameter is equal to PI . Hence, *mn* the locus of C is a hyperbola, and from what has been shewn every thing relative to it may be readily determined.

Next let us suppose that the weight b is placed at B (fig. 69. pl. 3.), upon HK an arch of a conic section, and that the pulley P is at the focus. It is easy to see that we may assume such a ratio for the given ratio of b to a , or of PB to BG , that MN (the locus of the point G) shall be the directrix of the section, and in this case the locus of the centre of gravity of the weights is a straight line. To determine the ratio of a to b , in this case, let γ denote the angle which AP makes with the directrix, and suppose the semi-transverse axis to be to the excentricity as 1 to e , then, drawing BL perpendicular to the directrix, we have BG

$= \frac{BL}{\sin. \gamma}$, and since by hypothesis $b : a :: PB : BG$, and from the nature of a conic section $PB : BL :: e : 1$, therefore $a : b :: 1 : e \sin. \gamma$. If a have to b any other ratio than that of 1 to $e \sin. \gamma$ the locus of G and consequently of C the centre of gravity is no longer a straight line but a conic section, as it is very easy to shew.

It is probable that the question, which we have just now resolved, was first suggested by this other problem. The same things being supposed as in the question, with this difference that one of the weights, instead of being placed upon an inclined plane, rests upon a curve of any kind. It is required to determine the position of the weights when they rest in equilibrio. To this problem the preceding analysis points out an easy solution. For draw GL (fig. 70. pl. 3.) perpendicular to PA . Then, because the weights will be in equilibrio when C their centre of gravity is as low as it can descend, and because when C is the lowest possible it appears from what has been shewn that G will also be
the

the lowest possible; it is evident the weights will be in equilibrio when PL is a maximum or minimum. Draw BQ perpendicular to PA; Put $PQ = x$, $QB = y$; then, because $b : a :: PB : BG$, we have BG or QL equal to $\frac{a}{b} \sqrt{(x^2 + y^2)}$, and $PL = \frac{a}{b} \sqrt{(x^2 + y^2)} - x$, a maximum or minimum; hence, taking the fluxions, &c. we find

$$\frac{a}{b} \left\{ x + \frac{\dot{y}y}{\dot{x}} \right\} = \sqrt{(x^2 + y^2)}.$$

Draw BR a normal to the curve HK, then it is manifest that $QR = \frac{\dot{y}y}{\dot{x}}$ and $PR = x + \frac{\dot{y}y}{\dot{x}}$, now $PB = \sqrt{(x^2 + y^2)}$, therefore the weights will be in equilibrio, whatever be the nature of the curve HK, when $\frac{a}{b} \times PR = PB$, that is, when

$$PB : PR :: a : b.$$

This conclusion agrees, as it ought to do, with what may be otherwise found by the resolution of forces.

SECOND SOLUTION, by Mr. LOWRY.

Fig. 71, pl. 3. Let AB be the inclined plane, C the position of the pulley, G the centre of gravity of the given weights P, Q, and l the length of the string. Draw GE parallel to the vertical line CP and CR perpendicular to the plane AB meeting EF, a parallel to AB, in F; then because the relative position of the pulley and plane is given, the line CR is given; and by the property of the centre of gravity, as the weight P : the weight Q :: GQ : GP, that is as EQ : EC :: RF : FC; therefore RF and FC are given lines. Again, as the sum of the weights P, Q is to the weight P :: PQ : GQ :: PC + CQ = l :

$GE + EQ = \frac{P}{P + Q} \times l$ a given line. On a line drawn

through F, parallel to CP, take $FL = \frac{P}{P + Q} \times l$, then L is a given point. Draw GS parallel and EK perpendicular to the plane AB, and take LH = to EK or FR; then because FL is = GE + EQ, and FS = EG, LS is = to EQ; and by similar triangles,

EF

$EF = SG : KQ :: FC : EK = LH$,
 or $SG^2 : KQ^2 = QE^2 - KE^2 = LS^2 - LH^2 :: FC^2 : LH^2$,
 which is the property of a hyperbola passing through H and having its centre at L, the lines HL and LI (a parallel to SG and $\equiv$ to FC) being two semi-conjugate diameters.

When the weights are equal, and the plane is horizontal, the hyperbola is equilateral. And when the pulley is placed at A the locus is then a straight line.

THIRD SOLUTION, by Mr. CUNLIFFE.

Let P be the pulley (fig. 72, pl. 3), AB the inclined plane, m and n the weights placed in any assigned position, G their centre of gravity, and l the length of the string. Draw GO parallel to AB and put $PA = a$, the sine and cosine of the angle $RAB = s$ and c , $AO = x$, $OG = y$, and the sum of the weights m and $n = w$. Then, by the property of the centre of gravity,

$$m : n :: Gn : Gm :: AC : Cm = \frac{n}{m} x,$$

$$\text{herefore } Pm = a + x + \frac{n}{m} x = a + \frac{w}{m} x.$$

$$\text{Again, } w : n :: Gn + Gm : Gm :: An : OG,$$

$$\text{or } An = \frac{w}{n} y; \text{ and by trigonometry,}$$

$$Pn = \sqrt{(AP^2 + An^2 + 2cAP \cdot An)} = \sqrt{(a^2 + \frac{w^2}{n^2} y^2 + \frac{2caw}{n} y)};$$

$$\text{therefore } Pn + Pm = \sqrt{(a^2 + \frac{w^2}{n^2} y^2 + \frac{2caw}{n} y)} + a + \frac{w}{m} x = l,$$

$$\text{or } a^2 + \frac{w^2}{n^2} y^2 + \frac{2caw}{n} y = (l - a - \frac{w}{m} x)^2;$$

and by reducing, we have

$$y^2 + \frac{2nca}{w} y - \frac{n^2}{m^2} x^2 + \frac{2n^2}{mw} (l - a) x - \frac{n^2}{w^2} (l^2 - 2al) = 0,$$

which is an equation to a hyperbola.

To determine the position of the curve, assume any point C for its centre, and draw CD parallel to mP meeting GO, BA produced in I and D, and the curve in L. Put $AD = d$, $DC = b$, the semi-diameter $CL = t$, and the parameter to that diameter $= p$; then $IG = d + y$ and $CI = b - x$, and by the property of the hyperbola,

$IG^2 : CI^2 - CL^2 :: p : 2t$,
 or $(d+y)^2 : (c-x)^2 - t^2 :: p : 2t$; which analogy being reduced to an equation, gives

$$2^2 + 2dy - \frac{p}{2t}x + \frac{p}{t}cx + d^2 + \frac{p}{2t}(t^2 - c^2) = 0.$$

Comparing this equation with the given one, we have $2d = \frac{ync a}{w}$, or $d = \frac{nca}{w}$; $\frac{p}{2t} = \frac{n^2}{m^2}$; $\frac{p}{t}c = \frac{2n^2}{m^2}c = \frac{2n^2}{mw}(l-a)$, or $c = \frac{m}{w}(l-a)$, and $d^2 + \frac{p}{2t}(t^2 - c^2) = -\frac{n^2}{w^2}(l^2 - 2al)$, or $t^2 = c^2 - \frac{m^2}{w^2}(l^2 - 2al) - \frac{m^2}{n^2}d^2 = \frac{m^2}{w^2}a^2(1 - c^2)$, (putting for c and d their values found above), and consequently $t = \frac{m}{w}as$. Also, since $\frac{p}{2t} = \frac{n^2}{m^2}$, we have $p = \frac{2n^2}{m^2} \times t = \frac{2n^2}{mw} \times sa$.

Now, let AD be taken $=$ to $\frac{n}{w}ca$, and on a line drawn parallel to Pm, take $OC = \frac{m}{w}(l-a)$, and $CL = \frac{m}{w} \times sa$; then C is the centre of the hyperbola, CL a semi-diameter, and $\frac{2n^2}{mw} \times sa$ the parameter of that diameter.

When a is $=$ 0, the equation becomes,

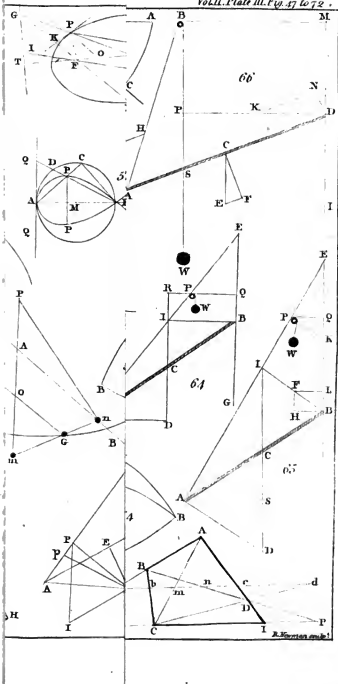
$$y^2 - \frac{n^2}{m^2}x^2 + \frac{2n^2}{mw}lx - \frac{n^2}{w^2}l^2 = 0,$$

$$\text{or } y^2 = \frac{n^2}{m^2}x^2 - \frac{2n^2}{mw}lx + \frac{n^2}{w^2}l^2,$$

and $y = \pm \left(\frac{n}{m}x - \frac{n}{w}l \right)$ which is an equation to a straight line.

When y is $=$ 0, x is $= \frac{m}{w}l$, and when x is $=$ 0, y is $= \frac{n}{w}l$; therefore $\frac{m}{w}l$ and $\frac{n}{w}l$ are the distances from the point A, where the locus meets the line Am and the plane AB.

The Rev. J. Toplis also sent a solution to this question.





XVII. QUESTION 167, *by LIMENUS.*

When any number of circles pass through the same two points, tangents drawn from any point in either of the two external circumferences to the remaining circles shall be to one another in a given ratio?

SOLUTION, *by* ———.

Let there be any number of circles PQA, PQB, PQC, PQD, &c. (fig. 73. pl. 4.) passing through the two points P, Q: then, if from any point A in the exterior segment PQA, tangents be drawn to the other circles they will have a given ratio. Let a, b, c, d , &c. be the centres of the given circles PQA, PQB, PQC, PQD, &c. which will evidently be in the same straight line; through the points A, Q, draw a line to meet the circles in B, C, D, &c. and draw PA, PB, PC, PD, &c. and Pa, Pb, Pc, Pd, &c. then the angles PAQ, PBQ, PCQ, PDQ, &c. being respectively equal to the angles Paq, Pbq, Pcq, Pdq, &c. the triangles PAB, PAC, PAD, &c. are respectively similar to the triangles Pab, Pac, Pad, &c. and consequently the line AD is divided similarly to the line ad , or the lines AB, AC, AD, &c. have the same ratio as the lines ab, ac, ad , &c. But the squares of the tangents drawn from A to the circles, are equal to the rectangles AQ·AB, AQ·AC, AQ·AD, &c. or proportional to the lines AB, AC, AD, &c. or to the lines ab, ac, ad , &c. as is proved above; and therefore the tangents are in the subduplicate ratio of the given distances ab, ac, ad , &c.

XVIII. QUESTION 168, *by LAPUTIENSIS.*

Suppose two bodies, whose weights are known to be connected by a flexible line or string, passing through a small hole in a perfectly smooth horizontal plane, and let the body which is above the plane be projected thereon, with a given velocity at a given distance from the hole, in a direction perpendicular to the string which is kept tight by the other body hanging freely; it is required to determine the velocity, time, and angle formed by the string with the first position, when the projected body has attained any other possible assigned distance from the hole through which the string passes?

SOLUTION,

SOLUTION, *by Mr. CUNLIFFE.*

Let A and B (fig. 74; pl. 4) represent the two bodies, the body B being that which is projected upon the horizontal plane, and the body A that which hangs freely below the plane by means of the connecting string BCA passing through the hole at C.

Let the curve BB'*b* represent the track of the body B, B' and *b* being two points therein indefinitely near to each other: draw CB', C*b* and with the centre C and radius CB' describe the small arch B'*d* cutting C*b* in *d*.

Put CB = *a*, CB' = *s*, db = *z*, *z* = circular arc (radius CB = *a*) intercepted between CB, CB'; *v* = the velocity at B' in the direction B'C or CB'; *t* = time of describing BB', and *g* = 32½; also let *f* denote the projectile velocity at B perpendicular to CB. By the property of tangents, we shall have B'*b* : B'*d* :: B'C : B'C × B*d* ÷ B'*b* = a perpendicular from C upon a tangent to the curve at B', and *bd* : B'*b* :: *v* : B'*b* × *v* ÷ *bd* = the velocity at B' in the direction of the curve.

Now, it is well known, that the velocities of a revolving body in different points of its orbit, are inversely as perpendiculars from the centre of force upon tangents to the orbit at those points;

wherefore $f : \frac{B'b \times v}{bd} :: \frac{B'C \times B'd}{B'b} : BC$, whence

$$\frac{B'd \times v}{bd} = \frac{BC \times f}{B'C} = \frac{af}{s}. \text{ It is manifest that } \frac{B'd \times v}{bd}$$

expresses the velocity of B' in the direction B'*d*, or the circulatory velocity at B': and hence the centrifugal force at B' is expressed

by $\frac{a^2 f^2}{s^3}$, and therefore the centrifugal force of the body B at that

point will be $\frac{a^2 f^2}{s^3} \times B$; from whence, deducting *Ag*, the gravity

of A, leaves $\frac{a^2 f^2}{s^3} \times B - Ag$, for the motive force urging both

bodies in the direction CB' or B'C; and consequently $\left(\frac{a^2 f^2}{s^3} \times B \right.$

$\left. - Ag \right) \div (A + B)$ will express the accelerative force in the same direction. Then, by a known theorem,

$$\left\{ \left(\frac{a^2 f^2}{s^3} \times B - Ag \right) \div (A + B) \right\} \times s = \ddot{v},$$

and

and taking the fluents,

$$-\left(\frac{a^2 f^2}{2s^3} \times B + Ags\right) \div (A+B) = \frac{v^2}{2};$$

but when $s = a$, $v = 0$, therefore the correct equation of the fluents is,

$$\left(\frac{f^2 \times B}{2} + Ags - \frac{a^2 f^2}{2s^3} \times B - Ags\right) \div (A+B) = \frac{v^2}{2}$$

$$\text{or, } v^2 = \frac{a-s}{s^3} \cdot \left\{ \frac{2Agss^2}{A+B} - \frac{Bf^2(a+s)}{A+B} \right\} = \frac{a-s}{s^2} \cdot \left\{ 2mgs^2 - f^2(1-m)(a+s) \right\}$$

$$\text{by putting } m = \frac{A}{A+B}, \text{ or } \frac{v^2}{2mg} = \frac{a-s}{s^2} \times \left\{ s^2 - \frac{f^2(1-m)(a+s)}{2mg} \right\}.$$

It may be remarked that when $g = 0$, the preceding expression for the velocity becomes $f^2 \times \left(\frac{s^2 - a^2}{s^2}\right) \times \frac{B}{A+B}$ or $v^2 \times \frac{A+B}{B} = f^2 \times \left(\frac{s^2 - a^2}{s^2}\right)$ which is the very expression

for the velocity obtained in the solution to question 105, vol. 1, and is therefore a confirmation of the truth of the conclusions, both in that and the present problem.

$$\text{We shall, moreover, have } \dot{z} = \frac{ss\sqrt{2mg}}{\sqrt{(a-s) \cdot \left\{ s^2 - \frac{f^2(1-m)(a+s)}{2mg} \right\}}}$$

the fluent of which may be had by infinite series, and correcting the same, by making the resulting expression to vanish when $s = a$.

The angle BCB' may be determined from the equation $\frac{B'd \times v}{bd} = \frac{af}{s}$ before deduced.

$$\text{For, from thence } B'd = \frac{af\dot{s}}{sv}; \text{ also } a : \dot{z} :: s : \frac{s\dot{z}}{a} = B'd,$$

$$\text{therefore } \frac{s\dot{z}}{a} = \frac{af\dot{s}}{sv} \text{ and } \dot{z} = \frac{a^2 f \dot{s}}{s^2 v} =$$

$$\frac{a^2 f \dot{s} \sqrt{2mg}}{\sqrt{(a-s) \times \left\{ s^2 - \frac{f^2(1-m)(a+s)}{2mg} \right\}}} \text{ the fluent of which}$$

may

may also be had by infinite series, which is to be corrected by making it to vanish when $s = a$, as observed in the preceding case.

A solution to question 105, vol. 1, might have been concisely effected as follows: We have determined that the motive centrifugal force at B', urging both bodies in the direction CB', is expressed by $\frac{a^2 f^2}{s^3} \times B$, and therefore the accelerative force is

$\frac{a^2 f^2}{s^3} \times \frac{B}{A+B}$; and hence, by a known theorem, $\frac{a^2 f^2}{s^3} \times \frac{B}{A+B} = v^2$, the fluent of which is $-\frac{a^2 f^2}{2s^2} \times \frac{B}{A+B} = \frac{v^2}{2}$, or $-\frac{a^2 f^2}{s^2} \times \frac{B}{A+B} = v^2$; but when $s = a$, $v = 0$, therefore the correct equation of the fluents is

$$\left(f^2 - \frac{a^2 f^2}{s^2}\right) \times \frac{B}{A+B} = f^2 \times \left(\frac{s^2 - a^2}{s^3}\right) \times \frac{B}{A+B} = v^2.$$

And thus the Question was answered by Mr. Bazley.

SECOND SOLUTION, by Mr. LOWRY.

Let M be the weight (or mass) of the projected body, and m that of the other body: also let C (fig. 75, pl. 4.) be the hole through which the string passes, A the position of the projected body at the commencement of the motion, and M its position at the end of any indefinite time t , AM $\hat{b}$ being a portion of the curve described by the body on the horizontal plane. Put $a = CA$, the initial distance of the body from the hole; b the velocity of projection, or the space described by the projected body in a second of time; $r = CM$ the radius vector of the revolving body, and $g = 32\frac{1}{2}$ feet the measure of the accelerative force of gravity. Then the projected body is acted upon by two forces, that is, by a motive force $= mg$, acting in the direction of the string MC and urging the body towards the centre C; and by a motive force Mf (f being the centrifugal force at M) acting also in the direction of the string but urging the body to recede from the centre; therefore the whole motive force acting on the mass M in the direction of the string is $= Mf - mg$. To determine the force f , let the point b be another position of the body indefinitely near to M; join C $\hat{b}$, and with the distance CM, and centre C, describe the arc

arc Ma . Draw CB perpendicular to the tangent at M , let u be the velocity of the body at M in the curve Mb , and w its velocity in the arc Ma . Then the triangles baM , ABM may be considered as similar, and $w : u :: aM : bM :: CB : CM = r$,

$$\text{or } w = \frac{u \times CB}{r}.$$

But the velocities of the revolving body at the points A , M are inversely as the perpendiculars on the tangents at those points, or as CB to CA ; therefore $u \times CB = b \times AC = ba$, and

$$\text{consequently } w = \frac{ba}{r}. \text{ Therefore } f \text{ is } = \frac{w^2}{r} = \text{to } \frac{b^2 a^2}{r^3};$$

and the whole motive force = to $\frac{Mb^2 a^2}{r^3} - mg$. Therefore the

$$\text{whole accelerative force is } = \frac{1}{M+m} \left(\frac{Mb^2 a^2}{r^3} - mg \right),$$

$$\text{and we have } \ddot{r} = \frac{\dot{r}^2}{M+m} \times \left(\frac{Mb^2 a^2}{r^3} - mg \right).$$

Multiply this equation by $2\dot{r}$,

$$\text{then } \dot{r} \ddot{r} = \dot{r}^2 \times \left(\frac{2Mb^2 a^2 \dot{r}}{r^3} - 2mg\dot{r} \right);$$

$$\text{and taking the fluents } \dot{r}^2 = \frac{\dot{r}^2}{M+m} \times \left(C - \frac{Mb^2 a^2}{r^2} - 2mgr \right).$$

Now at the commencement of the motion, or when $t = 0$, the quantity included between the parenthesis is also = to 0, and r being then = to a , it is obvious that C is = $Mb^2 + 2mga$;

$$\text{therefore } \dot{r}^2 = \frac{\dot{r}^2}{M+m} \times \left(Mb^2 + 2mga - \frac{Mb^2 a^2}{r^2} - 2mgr \right)$$

$$\text{and } \dot{r} = \frac{(M+m)^{\frac{1}{2}} \dot{r}}{\sqrt{\{ (Mb^2 + 2mga) r^2 - 2gmr^3 - Mb^2 a^2 \}}};$$

where t may be assigned by means of elliptic arcs.

$$\text{Again } \frac{\dot{r}}{\dot{t}} = \frac{1}{(M+m)^{\frac{1}{2}} r} \times \sqrt{\{ (Mb^2 + 2mga) r^2 - 2gmr^3 - Mb^2 a^2 \}}$$

is the velocity of the body in the direction of the string.

Let q be the angle that the string makes with the line from which the motion commences (reckoned on the arc of the circle whose

whose radius is 1), then $r : 1 :: \text{the arc } Ma = wt : \dot{q}$, or

$$\dot{q} = \frac{ba}{r^2} \dot{i} = \frac{(M+m)^{\frac{1}{2}} bar}{\sqrt{\{(Mb^2 + 2mga) r^2 - 2gmr^3 - Mb^2 a^2\}}}$$

The fluent, however, cannot be generally assigned without having recourse to an infinite series, or to some curve of a higher order than the conic sections.

XIX. QUESTION 169, by HERMODORUS.

Given the vanishing line, its centre and distance, to find the perspective representation of a circle from the given representation of one radius, or of an inscribed triangle?

SOLUTION, by 1550.

Let PQ (fig. 76, pl. 4.) be the given vanishing line, its centre O and its distance NO perpendicular to PQ, then from the other data it appears by Brook Taylor's Perspective, 8vo. 1715, Ex. 3 and 4, and Hamilton's Perspective, p. 80, that the representation of that diameter which tends to the centre O of the vanishing line may be found; let this be MK passing through I the centre of the circle, and let H be one of the points of the circle given in the question, and join MH, and KH meeting PQ in F and G, and draw HE parallel to PQ meeting OM in E.

Then since KM represents a diameter, KHM represents a right angle, therefore GNF is a right angle; also by similar triangles OM : ME :: OF : EH, and OK : KE :: OG : EH; therefore MO $\times$ OK : ME $\times$ EK :: ON² (= FO $\times$ OG) : EH², which is the property of a conic section, and from which, as MO $\times$ OK and ON² are given, it may be constructed by known methods.

Cor. By combining this solution with that in the 8th. Example of Brook Taylor's Perspective, 8vo. 1715, it is perfectly easy to find what is the representation of a sphere in any given position whatever and to construct the conic section.

N. B. There are three editions of Dr. Brook Taylor's Perspective, viz. in 1715, 1719, 1749, all in 8vo. and all scarce books; the two last are alike even to the press errors, but the first contains many things which the second does not, and the second that the first does not.

The

The reader may likewise consult Hamilton's *Perspective*, where he will find good constructions of these and several similar problems, as well as many other curious matters relating to *Perspective*: that book has no fault but its great prolixity, but it will amply reward the learned draftsman's patience by the complete knowledge of the theory and practice of *Perspective*, which he will learn from it.

XX. QUESTION 170, by HERMODORUS.

Given in position a circle and two points; if from the two points be inflected two straight lines to any point of the circle, and from the other two points in which these lines meet the circle, there be inflected two other straight lines to a point in the circle, so that one of them may be parallel to a straight line given by position, then the other straight line will either tend to a given point, or make a given angle with a straight line tending to a given point?

SOLUTION, by ———.

Let A, B (fig. 77, pl. 4.) be two given points, and CDE a circle given by position; if from the points A, B two straight lines AC, BC be inflected to any point C of the circle, meeting it in D and E; and from the points D, E two other straight lines DH, EH be inflected to any point H in the circle, so that one of them, EH, may be parallel to a straight line given by position; then the straight line DH, either tends to a given point, or makes a given angle with a straight line tending to a given point.

This is evident from the 57th. and 62d. Propositions of Dr. Simson's *Treatise de Porismatibus*, or which is the same, prob. iii. art. ix. vol. i, part ii. For, let the rectangle KAB be made equal to the given rectangle DAC, and then K will be a given point. Draw the diameter KIS, and divide it in F, so that $IF : FS :: IK : KS$, then F will also be a given point. Let DF be drawn to meet the circle in G and H. Then, by the 57th proposition, if the lines AC, BC be inflected to any point C of the circle, meeting it in D and E, the line joining the points D, E either passes through the point F, or makes a given angle at D with a straight line tending to that point, that is the angle EDG, or its equal the angle EHG, is given; and therefore since IE is parallel to a straight line given by position, GH is also parallel to a straight line given by position; therefore by the 62d. proposition, since a line DG is drawn through the given point F meeting the

circle in D and G, and GH is parallel to a straight line given by position, DH will either pass through a given point, or make a given angle at D with the straight line DK tending to the given point K, as is affirmed in the proposition.

The demonstration may be found in the propositions of Dr. Simfon already cited, or in the solution to the 129th question, Old Series, of the Repository.

XXI. QUESTION 171, *by* ANALYTICUS.

Find a series for the n th power of the logarithm of a number?

FIRST SOLUTION, *by* Mr. WALLACE.

Let $1+x$ denote any number, and X its hyperbolic logarithm, and put u for the series which is the expansion of X^n . Then, because $u = X^n$, we have $\dot{u} = nX^{n-1} \dot{X} = \frac{n \dot{X} X^{n-1}}{1+x}$; from these two equations we readily find

$$\frac{nu}{1+x} = \frac{X\dot{u}}{\dot{X}}.$$

Assume now $u = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c.$

Then, if it be considered that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c.$$

$$X = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

$$\frac{\dot{u}}{\dot{X}} = nx^{n-1} - (n+1)Ax^{n-2} + (n+2)Bx^{n-3} - (n+3)Cx^{n-4} + \&c.$$

by multiplying the series which expresses u and $\frac{1}{1+x}$, we get

$\frac{nu}{1+x}$ equal to

$$n \left\{ x^n - (1+A)x^{n+1} + (1+A+B)x^{n+2} - (1+A+B+C)x^{n+3} + \&c. \right\}$$

and by multiplying the series which expresses X and $\frac{\dot{u}}{\dot{X}}$, we get $\frac{X\dot{u}}{\dot{X}}$ equal to

$$nx^n - \left(\frac{n}{2} + \frac{n+1}{1}A \right) x^{n+1} + \left(\frac{n}{3} + \frac{n+1}{2}A + \frac{n+2}{1}B \right) x^{n+2}$$

$$-\left(\frac{n}{4} + \frac{n+1}{3}A + \frac{n+2}{2}B + \frac{n+3}{1}C\right)x^{n+3} + \&c.$$

By comparing the coefficients of like powers of x , in the two series, we obtain the following equations,

$$A = \frac{n}{2}$$

$$2B = \frac{2n}{3} + \frac{n-1}{2}A$$

$$3C = \frac{3n}{4} + \frac{2n-1}{3}A + \frac{n-2}{2}B$$

$$4D = \frac{4n}{5} + \frac{3n-1}{4}A + \frac{2n-2}{3}B + \frac{n-3}{2}C$$

$$5E = \frac{5n}{6} + \frac{4n-1}{5}A + \frac{3n-2}{4}B + \frac{2n-3}{3}C + \frac{n-4}{2}D$$

&c. &c.

the law of continuation being sufficiently obvious. From these the coefficients $A, B, C, D, \&c.$ may be readily determined for any given value of the exponent n .

The problem is here resolved, upon the supposition that the logarithm whose power is required is hyperbolic; but for any other system it is only necessary to multiply the series by the n th power of the modulus of the system.

SECOND SOLUTION, by Mr. IVORY.

By the binomial theorem $(1+a)^x =$

$$1 + x \cdot a + x \cdot \frac{x-1}{2} \cdot a^2 + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} \cdot a^3 + \&c. =$$

$$1 + x \cdot a + x(x-1) \cdot \frac{a^2}{2} + x(x-1) \left(\frac{x}{2} - 1\right) \cdot \frac{a^3}{3} + x(x-1) \left(\frac{x}{2} - 1\right) \left(\frac{x}{3} - 1\right) \cdot \frac{a^4}{4}$$

$$+ \&c. =$$

$$1 + x \cdot a - x(1-x) \frac{a^2}{2} + x(1-x) \left(1 - \frac{x}{2}\right) \frac{a^3}{3} - x(1-\frac{1}{2}x)(1-\frac{1}{3}x) \frac{a^4}{4} + \&c.$$

The *Terminus Generalis* of this series is,

$$\pm x(1-x)(1-\frac{1}{2}x)(1-\frac{1}{3}x)(1-\frac{1}{4}x) \dots \dots (1-\frac{1}{n}x) \times \frac{a^{n+1}}{n+1}$$

the upper or lower sign taking place according as the number

of binomial factors is even or odd. Let the notation $\left(\frac{n}{m}\right)$

be adopted to denote the aggregate sum of all the possible combinations, made by multiplying together m quantities out of these n quantities, viz. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$, so that, according to this notation, $\left(\frac{n}{1}\right)$ will denote the sum of these n fractions; and $\frac{n}{2}$ will denote the sum of all the products that arise by combining them by twos; and $\left(\frac{n}{3}\right)$ will denote the sum of all the products that arise by combining them by threes; and so on. Then, by what is commonly taught concerning the coefficients of binomial products in algebra, the *Terminus Generalis*, when developed and arranged according to the powers of x , will become

$$\pm \frac{a^{n+1}}{n+1} \cdot x \mp \left(\frac{n}{1}\right) \cdot \frac{a^{n+1}}{n+1} \cdot x^2 \pm \left(\frac{n}{2}\right) \cdot \frac{a^{n+1}}{n+1} \cdot x^3 \mp \left(\frac{n}{3}\right) \cdot \frac{a^{n+1}}{n+1} \cdot x^4$$

$\pm$ &c. the upper or lower sign taking place according as n is even or odd.

If, now, each term in the series that is equal to $(1+a)^x$ be expanded as we have just been explaining, and if all the terms be arranged according to the powers of x , we shall obtain,

$$\begin{aligned} (1+a)^x = 1 + x \times & \left\{ a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \&c. \right. \\ & + x^2 \times \left\{ \left(\frac{1}{1}\right) \cdot \frac{a^2}{2} - \left(\frac{2}{1}\right) \cdot \frac{a^3}{3} + \left(\frac{3}{1}\right) \cdot \frac{a^4}{4} - \&c. \right. \\ & + x^3 \times \left\{ \left(\frac{2}{1}\right) \cdot \frac{a^3}{3} - \left(\frac{3}{2}\right) \cdot \frac{a^4}{4} + \left(\frac{4}{2}\right) \cdot \frac{a^5}{5} - \&c. \right. \\ & + x^4 \times \left\{ \left(\frac{3}{2}\right) \cdot \frac{a^4}{4} - \left(\frac{4}{2}\right) \cdot \frac{a^5}{5} + \left(\frac{5}{2}\right) \cdot \frac{a^6}{6} - \&c. \right. \\ & \&c. \&c. \end{aligned}$$

The *Terminus Generalis* of this series being

$$+ x^n \times \left\{ \left(\frac{n-1}{n-1}\right) \cdot \frac{a^n}{n} - \left(\frac{n}{n-1}\right) \cdot \frac{a^{n+1}}{n+1} + \left(\frac{n+1}{n-1}\right) \cdot \frac{a^{n+2}}{n+2} - \&c. \right\}.$$

Let τ denote the hyp. log. of $1+a$, then $(1+a)^x$ (considered as a function of x) being expanded in the usual manner, we get

$$(1+a)^x = 1 + \frac{\tau}{1} \cdot x + \frac{\tau^2}{1 \cdot 2} \cdot x^2 + \frac{\tau^3}{1 \cdot 2 \cdot 3} \cdot x^3 + \frac{\tau^4}{1 \cdot 2 \cdot 3 \cdot 4} \cdot x^4 + \&c.$$

+ &c. the *Terminus Generalis* being $\frac{\tau^n}{1.2.3 \dots n} \propto x^n$.

But the coefficients of the same powers of x in the two developements must be equal to one another ; therefore

$$\frac{\tau^n}{1.2.3 \dots n} = \left(\frac{n-1}{n-1}\right) \cdot \frac{a^n}{n} - \left(\frac{n}{n-1}\right) \cdot \frac{a^{n+1}}{n+1} + \left(\frac{n+1}{n-1}\right) \cdot \frac{a^{n+2}}{n+2} - \&c.$$

and consequently τ^n , or the n^{th} power of the logarithm of $(1+a)$, =

$$1.2.3 \dots n \times \left\{ \left(\frac{n-1}{n-1}\right) \cdot \frac{a^n}{n} - \left(\frac{n}{n-1}\right) \cdot \frac{a^{n+1}}{n+1} + \left(\frac{n+1}{n-1}\right) \cdot \frac{a^{n+2}}{n+2} - \&c. \right\}$$

If we make $n=2$, then because the expressions $(\frac{1}{2})$, $(\frac{2}{2})$, $(\frac{3}{2})$, &c. denote the sums of one, two, and three, &c. of the quantities $1, \frac{1}{2}, \frac{1}{3}$, &c. we have the square of the logarithm of $(1+a)$ =

$$1.2 \times \left\{ \frac{a^2}{2} - (1 + \frac{1}{2}) \cdot \frac{a^3}{3} + (1 + \frac{1}{2} + \frac{1}{3}) \cdot \frac{a^4}{4} - \&c. \right\}.$$

And if we make $n=3$, then the cube of the logarithm of $(1+a)$ =

$$1.2.3 \times \left\{ \frac{1}{1.2} \cdot \frac{a^3}{3} - (\frac{2}{3}) \cdot \frac{a^4}{4} + (\frac{4}{3}) \cdot \frac{a^5}{5} - (\frac{4}{3}) \cdot \frac{a^6}{6} + \&c. \right\}$$

where the expressions $(\frac{2}{3})$, $(\frac{4}{3})$, $(\frac{4}{3})$, &c. denote the sum of the products of the couplets of three, and four, and five, &c. of the quantities $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c.

And if we make $n=4$, then the fourth power of the logarithm of $(1+a)$ =

$$1.2.3.4 \times \left\{ \frac{1}{1.2.3} \cdot \frac{a^4}{4} - (\frac{4}{3}) \cdot \frac{a^5}{5} + (\frac{4}{3}) \cdot \frac{a^6}{6} - (\frac{6}{3}) \cdot \frac{a^7}{7} + \&c. \right\}$$

where the expressions $(\frac{4}{3})$, $(\frac{4}{3})$, $(\frac{6}{3})$, &c. denote the sum of the products of all the triplets of four, and five, and six, &c. of the quantities $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, &c.

The method we have here used, will apply only when the index of the required power is a whole positive number. If the index be negative, or fractional, we may then have recourse to the ordinary method of assuming a series with indeterminate coefficients.

The Rev. Mr. Toplis likewise answered it.

XXII. QUESTION 172, *by* SENEX.

It is required to find the nature or equation of a curve such, that any right line RQP being drawn through a given point P, to cut the curve in two points R and Q, and tangents from these points produced to intersect each other in I, the angle QIR may always be of the same constant magnitude, that is always equal to a given angle?

FIRST SOLUTION, *by* Mr. IVORY.

Fig. 78, pl. 4. Let RT and QS be perpendicular to a fixed axis drawn through P; put VP, the distance of the point P from a given point V in the axis, $= b$, $VT = x$, $VS = x'$, $TR = y$, $SQ = y'$. Let the tangents RI and QI meet the axis in H and K: and let a = tangent of half the angle QIR (= sum of the angles RHT and QKS), and z = tangent of half the difference of the angles RHT and QKS: then tangent of

RHT $= \frac{a+z}{1-az}$, and tangent of QKS $= \frac{a-z}{1+az}$. But

$\frac{dy}{dx}$ = tangent of RHT, and $\frac{dy'}{dx'}$ = tangent of QKS. There-

fore $\frac{dx}{dy} = \frac{1-az}{a+z}$ and $\frac{dx'}{dy'} = \frac{1+az}{a-z}$. Again, because

the triangles TPR and QPS are equiangular, therefore $\frac{b-x}{y} = \frac{x'-b}{y'}$: and by taking fluxions,

$$\frac{dy}{y} \cdot \left\{ -\frac{dx}{dy} - \frac{b-x}{y} \right\} = \frac{dy'}{y'} \cdot \left\{ \frac{dx'}{dy'} - \frac{x'-b}{y'} \right\}.$$

Hence, observing that $\frac{b-x}{y} = \frac{x'-b}{y'}$, we get

$$\left(\frac{dy}{y} - \frac{dy'}{y'} \right) \cdot \frac{b-x}{y} = -\frac{dy}{y} \cdot \frac{dx}{dy} - \frac{dy'}{y'} \cdot \frac{dx'}{dy'},$$

$$\left(\frac{dy}{y} - \frac{dy'}{y'} \right) \cdot \frac{x'-b}{y'} = -\frac{dy}{y} \cdot \frac{dx}{dy} - \frac{dy'}{y'} \cdot \frac{dx'}{dy'}$$

or, which is the same thing,

$$2\left(\frac{dy}{y} - \frac{dy'}{y'}\right) \cdot \frac{b-x}{y} = -\left(\frac{dy}{y} + \frac{dy'}{y'}\right) \cdot \left(\frac{dx}{dy} + \frac{dx'}{dy'}\right) - \left(\frac{dy}{y} - \frac{dy'}{y'}\right) \cdot \left(\frac{dx}{dy} - \frac{dx'}{dy'}\right),$$

$$2\left(\frac{dy}{y} - \frac{dy'}{y'}\right) \cdot \frac{x'-b}{y'} = -\left(\frac{dy}{y} + \frac{dy'}{y'}\right) \cdot \left(\frac{dx}{dy} + \frac{dx'}{dy'}\right) - \left(\frac{dy}{y} - \frac{dy'}{y'}\right) \cdot \left(\frac{dx}{dy} - \frac{dx'}{dy'}\right),$$

$$\text{But } \frac{dx}{dy} + \frac{dx'}{dy'} = \frac{2a(1+z^2)}{a^2-z^2}, \text{ and } \frac{dx}{dy} - \frac{dx'}{dy'} = \frac{-2z(1+a^2)}{a^2-z^2};$$

therefore, if we put $\frac{dy'}{y} + \frac{dy}{y'} = \left(\frac{dy}{y} - \frac{dy'}{y'}\right) \times u$, we shall get

$$\frac{b-x}{y} = -u \times \frac{a(1+z^2)}{a^2-z^2} + \frac{z(1+a^2)}{a^2-z^2};$$

$$\frac{x'-b}{y'} = -u \times \frac{a(1+z^2)}{a^2-z^2} + \frac{z(1+a^2)}{a^2-z^2};$$

whence, we derive,

$$x = b + uy \cdot \frac{a(1+z^2)}{a^2-z^2} - y \cdot \frac{z(1+a^2)}{a^2-z^2};$$

$$x' = b - uy' \cdot \frac{a(1+z^2)}{a^2-z^2} + y' \cdot \frac{z(1+a^2)}{a^2-z^2}.$$

And if we take the fluxions of these equations, and substitute $dy \times \frac{1-az}{a+z}$ for dx and $dy' \times \frac{1+az}{a-z}$ for dx' ; then after having properly ordered the terms, we shall get

$$\frac{dy}{y} = \frac{du}{1-u} + \frac{u}{1-u} \cdot \frac{(1+a^2)2zdz}{(1+z^2)(a^2-z^2)} - \frac{a^2+1}{a} \cdot \frac{(a^2+z^2)dz}{(1+u)(1+z^2)(a^2-z^2)}$$

$$\frac{dy'}{y'} = \frac{-du}{1+u} - \frac{u}{1+u} \cdot \frac{(1+a^2)2zdz}{(1+z^2)(a^2-z^2)} + \frac{a^2+1}{a} \cdot \frac{(a^2+z^2)dz}{(1+u)(1+z^2)(a^2-z^2)}.$$

These equations contain all the conditions of the question; for they include the equation $\frac{dy}{y} + \frac{dy'}{y'} = u \times \left(\frac{dy}{y} - \frac{dy'}{y'}\right)$,

and, by this means the function u is left indefinite or arbitrary. Because x' , y' and x , y are co-ordinates of the same curve, therefore the two former must satisfy the equation of the curve equally as the two latter do; which requires that x' and y' be respectively the same functions of $-z$, that x and y are of $+z$. For, when this is the case, it is obvious that, by exterminating z from the expressions x' and y' , there will be obtained the same equation between x' and y' , as is obtained by exterminating z from the expressions x and y . This consideration restricts the nature of the functions of z that must be substituted for u : for only such functions of z must be substituted for u as will make the expressions

expressions for y and y' like functions of $+z$ and $-z$. Without entering into further detail it will be sufficient to remark that the required conditions will be fulfilled; that is, the expressions for y and y' will be like functions of $+z$ and $-z$; if there be substituted for u , functions of this form, viz. $Az + Bz^3 + Cz^5 + \&c.$; or such functions of z as, when expanded into series, contain only the odd powers of z . Having thus ascertained the proper functions of z to be substituted for u , we have only to consider the expressions of y and x , viz.

$$\frac{dy}{y} = \frac{du}{1-u} + \frac{u}{1-u} \cdot \frac{(1+a^2) 2z dz}{(1+z^2)(a^2-z^2)} - \frac{1+a^2}{a} \cdot \frac{(a^2+z^2) dz}{(1-u)(1+z^2)(a^2-z^2)},$$

$$x = b + uy \times \frac{a(1+z^2)}{a^2-z^2} - y \times \frac{(1+a^2)z}{a^2-z^2}.$$

In looking for the simplest curves that will satisfy the question, we are naturally led to examine the case, when $u = 0$. It is manifest from what has been observed, that this supposition is permitted. Now, when $u = 0$, then $\frac{dy}{y} = -\frac{1+a^2}{a} \cdot$

$\frac{(a^2+z^2) dz}{(1+z^2)(a^2-z^2)}$: And, by resolving into parts, to prepare

for integrating, $\frac{dy}{y} = -\frac{a^2-1}{a} \cdot \frac{dz}{1+z^2} - \frac{2adz}{a^2-z^2}$: and

by putting $m = \frac{a^2-1}{a} = a - \frac{1}{a}$, and then integrating,

$\log. y = \log. c - m \times \phi + \log. \frac{a-z}{a+z}$, where $\phi =$ arch whose

tangent is z : therefore if e be the number whose hyp. log. is 1, then $y = c \cdot e^{-m\phi} \times \frac{a-z}{a+z}$. But, when $u = 0$, then

$x = b - y \times \frac{(1+a^2)z}{a^2-z^2} = b - c(1+a^2) \cdot e^{-m\phi} \times$

$\frac{z}{(a+z)^2} = b - \frac{c(1+a^2)}{4a} \cdot e^{-m\phi} \times \frac{4az}{(a+z)^2}$. Let τ

denote the limit of ϕ , or the arch whose tangent is a , and let ϵ be determined, so that x and y may vanish together: then $b =$

$\frac{c(1+a^2)}{4a} \cdot e^{-m\tau}$; and, by substitution, $x =$

$\frac{c(1+a^2)}{4a} \cdot e^{-m\tau} \times \left\{ 1 - e^{m(\tau-\phi)} \times \frac{4az}{(a+z)^2} \right\}$; and,

by

by changing b to denote $\frac{c(1+a^2)}{4a} \cdot e^{-m\tau}$, the expressions for the co-ordinates of the curve become, $x = b - be^{m(\tau - \phi)}$
 $\times \frac{4az}{(a+z)^2}$, and $y = \frac{4ab}{1+a^2} \cdot e^{m(\tau - \phi)} \times \frac{a-z}{a+z}$;
 where a is the tangent of half the angle τ made by the tangents of the curve; $m = a - \frac{1}{a}$; tangent $\tau = a$; tangent $\phi = z$, the limits of ϕ being $\pm \tau$, and of z , $\pm a$.

If $a = 1$ (in which case the angle made by the tangents is a right-angle) then $m = 0$, and the expressions of the co-ordinates become, $x = b - b \cdot \frac{4z}{(1+z)^2}$ and $y = 2b \cdot \frac{1-z}{1+z}$; therefore, in this case the curve is algebraical; and if z be exterminated, we get $y^2 = 4bx$, the equation of a conic parabola having the parameter of the axis $= 4b$. The result of the analysis, in this case, when expressed in words, is this; "that tangents of a parabola drawn from the extremities of any line passing through the focus, intersect one another at right angles."

The supposition, next to that of $u = 0$, in point of simplicity, is to make $u = pz$. This substitution being made in the expressions of $\frac{dy}{y}$ and x , we get

$$\frac{dy}{y} = \frac{p \cdot dz}{1-pz} + \frac{1+a^2}{a} \cdot \frac{(2paz^2 - a^2 - z^2) \cdot dz}{(1-pz)(1+z^2)(a^2 - z^2)},$$

$$x = b + y \times \frac{(pa - 1 - a^2 + pa \times z^2) z}{a^2 - z^2}$$

It would be easy to examine if these new formulas comprehend any algebraic curves that deserve to be remarked on account of their simplicity. But, waving this discussion, we shall be content

to notice the case, when $pa = 1$, or $p = \frac{1}{a}$; the formulas then become,

$$\frac{dy}{y} = \frac{dz}{a-z} - \frac{(1+a^2) dz}{(a-z)(1+z^2)} = \frac{-adz}{1+z^2} - \frac{zdz}{1+z^2}$$

$$x = b - y \times z.$$

And by integrating, $\log. y = \log. b + a \int \frac{-dz}{1+z^2} - \log. \sqrt{(1+z^2)}$; and if tangent $\phi = \frac{1}{z}$; then $y = b \times e^{a\phi} \times \sin. \phi$.

Whence $x = b - b \times e^{a\varphi} \cos. \varphi$. And if e denote PR, the distance of the point of the curve, of which x and y are the co-ordinates from P, then $e = \{y^2 + (b - x)^2\} = be^{a\varphi}$; the equation of the common logarithmic spiral.

SECOND SOLUTION, by Mr. WALLACE.

Let QBR (fig. 79, pl. 4.) be a curve having the property required, and let its nature be expressed by an equation expressing the relation between PQ, a straight line drawn from P to any point of the curve, and some function of the angle contained by PQ, and PA, a line given by position. Put φ for the angle APQ, and as the angle PQI must be expressible in some way or other by the angle APQ, let the angle PQI be denoted by $f\varphi$, where by $f\varphi$ is meant a function of φ , of a determinate form, but as yet unknown.

Now, if we suppose the angle PQI to revolve about the point P, (the line PR remaining at rest) till the point Q arrive at R, then the tangent QI will coincide with RK, that is, with the tangent IR produced; and, as by such revolution, the angle APQ will have changed its value from φ to $\varphi + \pi$, (here π is put for two right-angles) while the angle PQI has changed its value so as to become PRK, it follows that whatever function the angle PQI is of φ , the same function will the angle PRK be of $\varphi + \pi$; therefore the angle PRK will be $f(\varphi + \pi)$. Put α for the angle I, which by the question is to be a constant quantity, then, because the angle I is the difference of the angles PRK, PQI, we have

$$f(\varphi + \pi) - f\varphi = \alpha,$$

an equation resolvable by the theory of *finite differences*, and which, employing the notation of that method, may be otherwise expressed thus;

$$\Delta(f\varphi) = \alpha.$$

Hence, taking the integral, and observing that $\Delta\varphi = \pi$, we have

$$f\varphi = \frac{\alpha}{\pi} \varphi + F(\sin. 2\varphi, \cos. 2\varphi);$$

where by $F(\sin. 2\varphi, \cos. 2\varphi)$ is meant any function whatever of the sine and cosine of 2φ ; let this arbitrary function be denoted

by ψ , then, putting $n\pi$ for α , so that $\frac{\alpha}{\pi} = n$, we have

$$\angle PQI = f\varphi = n\varphi + \psi.$$

Draw

Draw Pq indefinitely near to PQ and qs perpendicular to PQ ; let PQ be denoted by r ; then, putting dr and $d\phi$ for the differentials or fluxions of r and ϕ , we have $Qs = \pm dr$, and $sq = rd\phi$; and since $\tan. \angle PQI = \frac{sq}{Qs}$, therefore $\tan. (n\phi + \psi) = \frac{rd\phi}{\pm dr}$, and hence

$$\frac{dr}{r} = \frac{\pm d\phi}{\tan. (n\phi + \psi)}.$$

Now, before we can integrate this equation we must give some determinate form to the arbitrary function ψ , and this being done, the result will have this form,

$$\frac{dr}{r} = \Phi d\phi,$$

where by Φ is meant some known function of ϕ , and hence, by integrating, we shall have an equation expressing the nature of the curve sought. And as there is no limit to the number of different forms which may be given to the arbitrary function ψ , and consequently to the number of corresponding forms of the function Φ , it is evident that there are innumerable curves having the property required by the question.

In giving different forms to the function ψ , it may happen that some of them are such as to render the equation $\frac{dr}{r} = \Phi d\phi$ capable of having its integral expressed in finite terms. This, however, will not be the case in general, and perhaps it may not be easy to assign such forms to the function ψ , as shall produce a finite integral.

If we suppose that $\psi = \beta$, a given angle, then the fluxional equation becomes,

$$\frac{dr}{r} = \frac{\pm d\phi}{\tan. (n\phi + \beta)} = \frac{\pm 1}{n} \cdot \frac{\text{cof. } (n\phi + \beta) d n\phi}{\sin. (n\phi + \beta)};$$

and hence, by integrating, and putting c for a constant quantity, we get

$$r = c \sin. \pm \frac{1}{n} (n\phi + \beta),$$

an equation which expresses the nature of an indefinite number of curves having the property required by the question.

Let us suppose α to be a right-angle, and assume $\beta = 0$, then, because in this case $n = \frac{\alpha}{\pi} = \frac{1}{2}$, we have

$$r = c \sin. \pm 2 \frac{1}{2} \phi.$$

If we give the negative sign to the exponent, we have

$$r = \frac{2c}{2 \sin. \frac{1}{2} \varphi} = \frac{2c}{1 - \cos. \varphi},$$

an equation which is known to belong to a parabola (fig. 79), having P for its focus, and PA for its axis.

If, again, we suppose the exponent to be positive, we have

$$r = c \sin. \frac{1}{2} \varphi = \frac{c}{2} (1 - \cos. \varphi).$$

This equation belongs to an epicycloid PQBR (fig. 80, pl. 4.) generated by a given point Q, in the circumference of a circle DQ, which revolves on the outside of another circle PD of equal magnitude, the given point Q, in the revolving circle being supposed to depart from P, which is also the point of contact of the two circles at the beginning of the motion, and the diameter of each being equal to the constant quantity c . For suppose that Q is a point in the curve thus generated; draw CDE through the centres of the circles and their point of contact, and draw QE, PF perpendicular to CD. Then, because $\text{arch PD} = \text{arch QD}$, the straight line PQ is parallel to CE, and $PQ = FE = \frac{1}{2} FD$. But because $CD = CP$, and $CF = CP \times \cos. PCF = PC \times -\cos. APQ$, therefore $PQ = CP \times (1 - \cos. APQ)$, that is, $r = \frac{c}{2} (1 - \cos. \varphi)$.

If we resume the equation

$$\frac{dr}{r} = \frac{\pm d\varphi}{\tan. (n\varphi + \psi)};$$

and suppose $\alpha = 0$, that is, if we suppose the tangents at the extremities of the line QR to be parallel, then, as in this case,

$n = \frac{\alpha}{\pi} = 0$, we have, (assuming $\psi = a$ constant angle, and putting m for its tangent)

$$\frac{dr}{r} = \pm \frac{1}{m} d\varphi,$$

hence, by integrating, and putting c for a constant quantity, we get

$$\log. r = c\varphi,$$

an equation to the logarithmic spiral.

XXIII. QUESTION 173, by Mr. JOHN FLETCHER.

If the magnitude of a circle is given; it is required to find the position of a right line RS, and that of a point P in another line given

given in position, from whence, if PAB be drawn to cut the circle in A and B, and perpendiculars AR and BS are drawn to RS, their rectangle will be always a given magnitude?

SOLUTION, by Mr. LOWRY.

This problem is easily resolved by means of the following lemma, which is the same as prop. xviii. of Dr. Stewart's Propositiones Geometricæ.

Fig. 81, 82, pl. 4. If two points C, P, be taken in the diameter EF of a circle, so that the square of PC may be equal to the rectangle ECF, and a line SCK be drawn perpendicular to PF; then if any straight line be drawn from P to meet the circle in A and B, and perpendiculars AR, BS, be drawn to the line SK, the rectangle of those perpendiculars will be equal to the square of PC.

Let PB meet SK in L, and draw LE to meet the circle in Q; then $ALB = ELQ = CL^2 + ECF = CL^2 + CP^2 = PL^2$; and by similar triangles,

$$AR : AL :: PC : PL,$$

$$BS : LB :: PC : PL;$$

$$\text{therefore } AR \cdot BS : ALB = PL^2 :: PC^2 : PL^2,$$

$$\text{and } AR \cdot BS = PC^2.$$

This being premised, let O be the centre of the circle, produce the diameter FE till the rectangle ECF be equal to the given space, or to the square of the line n ; and then the distance CO will be given. On FC produced take CP = to n , and then the distance CP will be given: with this distance and centre O, describe an arc to cut the given line in P, and then SCK, drawn through C, perpendicular to PO, is the line required.

For, by the lemma, the rectangle AR·BS is always equal to PC^2 or to the given space n^2 .

Mr. Cunliffe also answered this question.

XXIV. QUESTION 174, by AMICUS.

If a given cone, suspended by its vertex, be made to revolve with its axis in a vertical plane; it is required to determine the force exerted on the centre of suspension at any given position of the revolving body?

SOLUTION,

SOLUTION, by Mr. LOWRY.

Instead of confining the solution to the particular case of the cone, as proposed in the question, I shall resolve the problem generally, when the body is any homogeneous solid formed by the revolution of a plane figure about an axis.

Let ASB (fig. 83, pl. 4.) be a section of the body made by the vertical plane in which the axis SC oscillates or revolves; also, let G be the centre of gravity, O the centre of oscillation, and S the centre of suspension of the body. Let p be the measure of the angle (reckoned on the periphery of a circle whose radius is 1) that the axis makes with the horizontal line SF, when the body is in any assigned position; and put $SG = a$, $SO = b$, $g = 32\frac{1}{2}$ feet, the measure of the accelerative force of gravity, and w the weight of the body.

Then the force of gravity (or weight w) may be resolved into two forces, the one equal to $\sin. p \times w$ acting in the direction of the axis SC, and the other equal to $\cos. p \times w$ acting perpendicularly to SC. Now the force $\sin. p \times w$ acting in the direction SC, can have no effect on the motion of the body, but is wholly exerted on the centre of suspension, and, when reduced to the horizontal direction SF, it is equal to $\sin. p \times \cos. p \times w$.

Again, the force $\cos. p \times w$, acting perpendicularly to SC, is that which accelerates the motion of the body about the centre, supposing the whole mass of the body to be concentrated at O, but the absolute velocities produced at G and O being directly as the distances a , b , from the centre of motion, the velocity at

G is only the $\frac{a}{b}$ part of the velocity at O, and therefore the force which is employed at G, in accelerating the motion of the body is only the $\frac{a}{b}$ part of the force $\cos. p \times w$, therefore the

remaining part of this force, or $\frac{b-a}{b} \cos. p \times w$, is exerted on the axis of suspension in a direction perpendicular to SC. This part of the force, when reduced to the horizontal direction FS,

is = to $\frac{b-a}{b} \sin. p \cos. p \times w$, and as it acts in a contrary direction to the other horizontal force $\sin. p \cos. p \times w$, we have $(1 - \frac{b-a}{b}) \sin. p \cos. p \times w$, or $\frac{a}{b} \sin. p \cos. p \times w$, for the whole effect of gravity on the centre of suspension in the horizontal direction SF.

But

But this is not the whole of the force that acts upon the centre of suspension, for it is also affected by the centrifugal force of the body, and to compute the effect of this force, let f be the centrifugal force of a particle at O , and v its velocity; then the forces of an equal particle, at the distances $d, d', d'', \&c.$ from the axis of motion, are equal to $\frac{f}{b} \times d, \frac{f}{b} \times d', \frac{f}{b} \times d'' \&c.$ therefore the centrifugal force of the whole body is equal to $\frac{f}{b}$ multiplied by the sum of the products of each particle by its distance from the axis of motion, or (by the property of the centre of gravity) $=$ to $\frac{fa}{b}$ multiplied by the mass of the body.

Now, by the nature of central forces, the force f is equal to $\frac{v^2}{b}$, and to eliminate v , let a circle be described about S with the distance SO , meeting the vertical line that passes through S in D and E , and draw OI parallel to FS . Then, if the body revolves about the centre S , let HE be taken equal to the height due to the velocity of the body at the lowest point E ; but if the body only oscillates through a part of the circle, as through the arc EK , then let KH be drawn parallel to FS meeting the vertical line in H ; in either case HI will be the height due to the velocity of the body at O , or v^2 will be equal to $2g \times HI$. But $SI = b \times \sin. p$, and putting $SH = c$, we have $HI = b \times \sin. p \pm c$, and $v^2 = 2g (b \sin. p \pm c)$ the upper or lower sign taking place as the point H is above or below the centre; therefore $\frac{f}{b}$ is $= 2g \left(\frac{b \sin. p \pm c}{b^2} \right)$. But the weight of the body is equal to g multiplied by the mass, therefore $\frac{fa}{b}$ multiplied by the mass is equal to $w \times \frac{2a}{b^2} (b \sin. p \pm c)$, the whole centrifugal force of the body in the direction SO ; which being reduced to the horizontal direction SF is $= w \times \frac{2a \cos. p}{b^2} (b \sin. p \pm c)$.

Let this force be added to the force of gravity found above, and we have

$$w \times \frac{a}{b} (\sin. p \cos. p) + w \times \frac{2a \cos. p}{b^2} (b \sin. p \pm c), \text{ or}$$

$w \times$

$w \times \frac{a}{b} \left\{ (3 \sin. p \cos. p) \pm \frac{2c}{b} \cos. p \right\}$ for the whole force required, when estimated in a horizontal direction.

To find the value of the angle p when this force is the greatest, let the fluxion of the variable part of the above expression be taken, and we have

$$3 \sin. \dot{p} \cos. p + 3 \cos. \dot{p} \sin. p \pm \frac{2c}{b} \cos. \dot{p} = 0;$$

substitute for $\sin. \dot{p}$ and $\cos. \dot{p}$, their values $\dot{p} \cos. p$ and $-\dot{p} \sin. p$, and divide by $3\dot{p}$,

$$\text{then } \cos.^2 p - \sin.^2 p \mp \frac{2c}{3b} \sin. p = 0,$$

$$\text{or } \sin.^2 p \pm \frac{c}{3b} \sin. p = \frac{1}{2},$$

$$\text{and } \sin. p = \frac{\sqrt{(18b + c^2) \pm c}}{6b}.$$

The force in the direction SF, arising from gravity being reduced to the vertical direction SE, is evidently equal to $w \times \frac{a}{b} \sin.^2 p$; and the centrifugal force, when reduced to the same direction, is equal to $w \times \frac{a}{b} \sin. p (b \sin. p \pm c)$; and therefore the whole force acting on the centre of suspension in the vertical direction SE, is

$$= w \times \frac{a}{b} (3 \sin.^2 p \pm \frac{2c}{b} \sin. p).$$

When p is a quadrant, or the axis of the body coincides with the line SE, the force is $= w \times \frac{a}{b} (3 \pm \frac{2c}{b})$; and when the points G and O coincide (as in the case of a ball suspended by a string) and the body oscillates through a quadrant, c is then $= 0$, and the whole force $= 3w$, as determined in the solution to the 106th question.

In the case of the cone, if d be the altitude, and r the radius of the base; then a is $= \frac{1}{2}d$, and $b = \frac{4d^2 + r^2}{5d}$, as is well known; and the expression for the horizontal force, is

$w \times$

$w \times \frac{15d^2}{4(4d^2 + r^2)} (3 \sin. p \cos. p \pm \frac{10cd}{4d^2 + r^2} \cos. p)$, and for the vertical force $w \times \frac{15d^2}{4(4d^2 + r^2)} (3 \sin.^2 p \pm \frac{10cd}{4d^2 + r^2} \sin. p)$.

When the cone oscillates only through the arc EK, then, these expressions may be a little simplified, for since SH is the cosine of the arc EK, if q be put for the measure of the angle

ESK (to the radius 1), then SH or c is $= b \cos. q$, and $\frac{2c}{b} = 2 \cos. q$, and the above expressions for the forces become

$$w \times \frac{15d^2}{4(4d^2 + r^2)} (3 \sin. p \cos. p \pm 2 \cos. q \cos. p),$$

$$\text{and } w \times \frac{15d^2}{4d^2 + r^2} (3 \sin.^2 p \pm 2 \cos. q \sin. p).$$

XXV. QUESTION 175, by Mr. WALLACE, R. M. College.

Suppose a quadrilateral figure to be inscribed in another quadrilateral figure, agreeable to the conditions required in question 120 of the New Series of the Repository, then the opposite sides of the inscribed figure when produced, will intersect each other in the diagonals produced of the circumscribed figure. Required the demonstration?

SOLUTION, by Mr. WILLIAM SIMPSON, Bolton.

Let $abcd$ (fig. 84, pl. 4.) be a quadrilateral, inscribed in ABCD agreeably to the conditions of the question, and let the opposite sides ac , bd meet in g ; join Bg, Cg and draw the diagonal BC. Then it appears from the solution to question 120 that aB bisects the angle bag , that bB bisects the exterior angle abg , that dC bisects the exterior angle cdM , and that Cc bisects the exterior angle dcG ; therefore Bg bisects the angle agb , and Cg bisects the angle cgd , as is pretty generally known (see Emerson's Geometry 35, ii.). Therefore Bg and Cg bisect the same angle and consequently coincide, and CBg is a straight line. It is proved in the same manner that the other sides and diagonal when produced meet in a point.

SECOND SOLUTION, by Mr. J. JOHNSON.

Let $abcd$, fig. 85, pl. 4.) be a quadrilateral inscribed in another quadrilateral, $ABCD$, agreeably to the conditions required in question 120. Then, if the opposite sides ab , cd , when produced, intersect in the point H , it is to be proved that the diagonal AC , when produced, will also pass through that point.

From A demit the perpendiculars AQ , AP , AR ; then it appears from the solution to question 120, that Ad bisects the angle PdQ , and that Aa bisects the angle RaQ ; therefore the right-angled triangles APd , AQd are equiangular, and the side Ad is common to each triangle, consequently they are equal in all respects, and $AP = AQ$. For a like reason the triangles ARa , AQa are equal in all respects, or $AR = AQ$; therefore AP is $= AR$.

Again if the perpendiculars CE , CF , CG , be drawn from C , it is proved in the same way as above that CG is $= CF$; therefore $CG : CF :: AP : AR$; and consequently the points H , A , C are in a straight line.

It is proved in the same manner that the diagonal BD will pass through h , the point where the opposite sides bc , ad intersect when produced.

The demonstrations received from Messrs. Swale and Whitley, are little different from those above.

XXVI. QUESTION 176, by MARLOVIENSIS.

It is well known that the sum of the series

$$x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2} - \&c. \text{ ad infinitum, is ex-}$$

pressed by the fluent of $(x^{-1} \times \text{fl. } \frac{x}{1+x})$, or area of a trans-

cendent hyperbola. Under what circumstances can the same sum be exhibited by the multiples or powers of the arc of a circle, and how is this to be investigated?

SOLUTION, by Mr. IVORY.

The investigation of the sums of the powers of the reciprocals of the natural numbers seems first to have engaged the attention of the two brothers, James and John Bernoulli. In a short treatise

treatise on the sums of infinite series, written by James Bernoulli, ("Tractatus de seriebus infinitis, &c." published with the "Ars Conjectandi" of the same author in one volume) we are informed (prop. 14) that John Bernoulli, first of all demonstrated that the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ is greater than any finite number. In the same treatise, the sum of the infinite series,

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \&c. = \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \&c.$$

is shewn to be equal to $\frac{1}{2}$: and from hence it follows that the sum of the infinite series $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$ is a finite quantity less than $\frac{1}{2}$. But the summation of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$$

was a problem for the solution of which the artifices then known were insufficient: and James Bernoulli, speaking of this series, uses the following words, "difficilior est, quam quis expectaverit, summæ investigatio; quam tamen finitam esse, ex altera, qua manifesto minor est, colligimus: si quis inveniatur, nobisque communicet, quod industriam elusit hætenus, magnas de nobis gratias feret." The further discoveries of the Bernoullis, on this subject, are confined to the demonstration of the following properties; viz.

$$\left(1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \&c.\right) \times \frac{2^m - 1}{2^m} = 1 + \frac{1}{3^m} + \frac{1}{5^m} + \frac{1}{7^m} + \&c.$$

$$\left(1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \&c.\right) \times \frac{2^{m-1} - 1}{2^{m-1}} = 1 - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} + \&c.$$

The summation of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \&c.$ was accomplished by Euler; but the discovery was too late to gratify the curiosity of James Bernoulli, who was dead. The method of Euler extends to the general series $1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c.$ where the index is any even number: and it is derived from the expression obtained by resolving the series for the sine of an arch into its binomial factors. Let z be any arch, and π the semi-circumference to the radius 1; then $\sin. z$

$$= z - \frac{z^3}{2 \cdot 3} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c. = z \left(1 - \frac{z^2}{\pi^2}\right) \left(1 - \frac{z^2}{2^2 \pi^2}\right) \left(1 - \frac{z^2}{3^2 \pi^2}\right) \&c.$$

and, by taking the fluxions of the logarithms of both sides of this equation and dividing by dz ,

$$\frac{\text{cof. } z}{\text{fin. } z} = \frac{1}{z} - \frac{2z}{\pi^2 - z^2} - \frac{2z}{2^2\pi^2 - z^2} - \frac{2z}{3^2\pi^2 - z^2} - \&c. \text{ let } \frac{\text{cof. } z}{\text{fin. } z} =$$

$$\cot. z = \frac{1}{z} - A^{(1)} z - A^{(3)} z^3 - A^{(5)} z^5 - \&c. \text{ the general}$$

term of the developement being $-A^{(2n-1)} z^{2n-1}$; and let

the several fractions in the value of $\frac{\text{cof. } z}{\text{fin. } z}$ be developed

into series: then by equating the coefficients of the same powers of z , on different sides of the equation, we obtain,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = \frac{A^{(1)}}{2} \cdot \pi^2 = \frac{\pi^2}{6}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c. = \frac{A^{(3)}}{2} \cdot \pi^4 = \frac{\pi^4}{90}$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. = \frac{A^{(5)}}{2} \cdot \pi^6 = \frac{\pi^6}{945}$$

$$\text{and, in general, } 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. = \frac{A^{(2n-1)}}{2} \times \pi^{2n}.$$

From the value of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$ we immediately derive,

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \&c. = \frac{\pi^2}{12}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. = \frac{\pi^2}{8}$$

These summations, first obtained by Euler, have been investigated by other mathematicians in different ways; and, particularly, by Mr. Landen in various parts of his writings.

If we put $x = 1$, in the series proposed in the question, it becomes $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \&c.$ the sum of which is $\frac{\pi^2}{12}$, as we have seen; and thus one case is obtained in which the sum of that series is expressed by the square of an arc of a circle. But, if I mistake not, the ingenuity of mathematicians has hitherto discovered no other case, that of $x = 1$ excepted, in which the sum of the series $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \&c.$ can be exhibited by means of arcs of a circle alone, without logarithms.

rithms. Mr. Landen, indeed, has succeeded in determining the sum of the series $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c.$ and the sums of the kindred serieses $x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c.$ and $x + \frac{x^2}{3^2} + \frac{x^3}{5^2} + \&c.$ in some other particular instances by means of circular arcs and logarithms; and these cases I shall now proceed to notice, although they seem hardly to fall within the scope of the question.

For the sake of abridging language, I shall use the following notations: *viz.*

the symbol $F(x)$ for the series $x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c.$

the symbol $f(x)$ for the series $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c.$

and the symbol $\phi(x)$ for the series $x + \frac{x^2}{3^2} + \frac{x^3}{5^2} + \&c.$

Since $\int \frac{dx}{1+x} = x \pm \frac{x^2}{2} + \frac{x^3}{3} \pm \frac{x^4}{4} + \&c.$ therefore

$\int \frac{dx}{x} \int \frac{dx}{1+x} = x \pm \frac{x^2}{2^2} + \frac{x^3}{3^2} \pm \frac{x^4}{4^2} + \&c.$ these

formulas, which are alluded to in the question, belong to De Lorgna: by putting $\log. \frac{1}{1-x}$ for $\int \frac{dx}{1-x}$ and $\log. (1+x)$

for $\int \frac{dx}{1+x}$, and attending to the meaning of the symbols just explained, we get

$$\int \frac{dx}{x} \log. \frac{1}{1-x} = F(x)$$

$$* \quad \int \frac{dx}{x} \log. (1+x) = f(x)$$

and, adding these expressions,

$$\int \frac{dx}{x} \log. \frac{1+x}{1-x} = 2\phi(x).$$

Now, let $x+y=1$: then $y=1-x$; and $-\int \frac{dx}{x} \log. y = -\int \frac{dx}{x} \log. (1-x) = F(x)$: in like manner, $-\int \frac{dy}{y} \log. x = F(y)$; therefore, $F(x) + F(y) = -\int \left\{ \frac{dx}{x} \log. y + \frac{dy}{y} \log. x \right\} = C - \log. x \times \log. y$: to determine C , let $x=0$, then $F(x)$

$F(x) = 0$, $\log. x \times \log. y = 0$, and $F(y) = F(1) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \&c. = \frac{\pi^2}{6}$: therefore,

$$F(x) + F(1-x) = \frac{\pi^2}{6} - \log. x \times \log. (1-x) \dots \dots (1).$$

Let $x = 1-x$, or $x = (\frac{1}{2})$, then $F(\frac{1}{2}) = \frac{1}{2} + \frac{1}{2^2} (\frac{1}{2})^2 + \frac{1}{3^2} (\frac{1}{2})^3 + \frac{1}{4^2} (\frac{1}{2})^4 + \&c. = \frac{\pi^2}{12} - \frac{1}{2} \log.^2 2$.

Again, let $y = \frac{x}{1+x}$: then $\frac{dy}{y} = \frac{dx}{x} - \frac{dx}{1+x}$, and $1-y = \frac{1}{1+x}$; therefore $\frac{dy}{y} \log. \frac{1}{1-y} = \frac{dx}{x} \log. (1+x) - \frac{dx}{1+x} \log. (1+x)$; and, by integrating $F(y) = f(x) - \frac{1}{2} \log.^2 (1+x)$, that is,

$$f(x) - F\left(\frac{x}{1+x}\right) = \frac{1}{2} \log.^2 (1+x) \dots \dots (2).$$

By the formula (1), we get $F\left(\frac{1}{1+x}\right) + F\left(\frac{x}{1+x}\right) = \frac{\pi^2}{6} - \log. \frac{1}{1+x} \times \log. \frac{x}{1+x}$: and by combining this equation with the formula (2), we get

$$f(x) + F\left(\frac{1}{1+x}\right) = \frac{\pi^2}{6} - \frac{1}{2} \log.^2 (1+x) + \log. (1+x) \times \log. x \dots (3).$$

Let $a = \frac{1}{1+x}$; then $a^2 + a = 1$, and $a = \frac{\sqrt{5}-1}{2}$: put a for x in the formula (3), then

$$f(a) + F(a) = \frac{\pi^2}{6} - \frac{1}{2} \log.^2 a.$$

Put a for x in the formula (2), and because $a^2 = \frac{a}{1+a}$, therefore

$$f(a) - F(a^2) = \frac{1}{2} \log.^2 a.$$

But, from the nature of the functions $f(a)$ and $F(a)$, it is obvious that we have $F(a) - \frac{1}{2} F(a^2) = f(a)$; and thus three equations are obtained that are sufficient to determine the quantities $F(a)$, $f(a)$, and $F(a^2)$; put $b = a^2 = \frac{3-\sqrt{5}}{2}$; and we thus get

$$F(a)$$

$$F(a) = a + \frac{a^3}{2^3} + \frac{a^5}{3^3} + \&c. = \frac{\pi^2}{10} - \log.^2 a$$

$$f(a) = a - \frac{a^3}{2^3} + \frac{a^5}{3^3} - \&c. = \frac{\pi^2}{15} - \frac{1}{2} \log.^2 a$$

$$F(a^2), \text{ or } F(b) = b + \frac{b^3}{2^3} + \frac{b^5}{3^3} + \&c. = \frac{\pi^2}{15} - \frac{1}{2} \log.^2 b$$

$$\text{where } a = \frac{\sqrt{5}-1}{2} \text{ and } b = \frac{3-\sqrt{5}}{2}.$$

Suppose $y = \frac{1-x}{1+x}$: then $-\int \frac{dx}{x} \log. y = \int \frac{dx}{x} \log.$
 $\frac{1+x}{1-x} = 2\phi(x)$: and because $x = \frac{1-y}{1+y}$, therefore, in
 like manner, $-\int \frac{dy}{y} \log. x = 2\phi(y)$: therefore,

$$\phi(x) + \phi(y) = -\frac{1}{2} \int \left\{ \frac{dx}{x} \log. y + \frac{dy}{y} \log. x \right\} = c - \frac{1}{2} \log. x \times \log. y:$$

but when $x = 0$, then $\phi(x) = 0$, $\log. x \times \log. y = 0$, and $\phi(y)$
 $= \phi(1) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{\pi^2}{8} = c$: therefore,

$$\phi(x) + \phi\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{8} - \frac{1}{2} \log. x \times \log. \frac{1-x}{1+x} \dots (4)$$

let $e = \frac{1-e}{1+e}$; then $e^2 - 2e = 1$, and $e = \sqrt{2} - 1$: and
 by putting $x = e$, the formula (4) gives

$$\phi(e) = e + \frac{e^3}{3^2} + \frac{e^5}{5^2} + \&c. = \frac{\pi^2}{16} - \frac{1}{2} \log.^2 e.$$

Again, a being $= \frac{\sqrt{5}-1}{2}$, we have obtained above, $f(a)$
 $+ F(a) = \frac{\pi^2}{6} - \frac{1}{2} \log.^2 a$: but $f(a) + F(a) = 2\phi(a)$:
 therefore,

$$\phi(a) = a + \frac{a^3}{3^2} + \frac{a^5}{5^2} + \&c. = \frac{\pi^2}{12} - \frac{1}{2} \log.^2 a.$$

Lastly, let $d = \frac{1-a}{1+a} = \sqrt{5}-2$: then, by substituting
 the value of $\phi(a)$ just obtained, in the formula (4), we obtain
 $\phi(d) = d + \frac{d^3}{3^2} + \frac{d^5}{5^2} + \&c. = \frac{\pi^2}{24} + \frac{1}{2} \log.^2 a - \frac{1}{2} \log. a \times \log. d.$

The

The formulas (2) and (3) deserve notice, because the computation of $F(x)$, when x is less than 1, is, by them, reduced to the computation of $f(x)$, when x is less than 1. And, because

$$x = \frac{1 + \frac{x}{1-x}}{1 - \frac{x}{1-x}}; \text{ therefore } \log. x = \log. (1+x) - \log. \left(1 + \frac{1}{x}\right):$$

$$\text{therefore } \frac{dx}{x} \log. (1-x) - \frac{dx}{x} \log. \left(1 + \frac{1}{x}\right) = \frac{dx}{x} \log. x:$$

therefore $f(x) + f\left(\frac{1}{x}\right) = C + \frac{1}{2} \log.^2 x$: to determine C put

$x = 1$, then $C = 2f(1) = \frac{\pi^2}{6}$; therefore,

$$f(x) + f\left(\frac{1}{x}\right) = \frac{\pi^2}{6} + \frac{1}{2} \log.^2 x \dots \dots \dots (5).$$

By this formula, the computation of $f(x)$ when x is greater than 1, is reduced to the computation of $f(x)$ when x is less than 1. Thus the computation of $F(x)$ when x is less than 1, and of $f(x)$ in all cases whatsoever, is reduced to the computation of $f(x)$ when x is less than 1. Now this may be accomplished by a converging series as follows:

Let $z = \log. \sqrt{1+x}$, and let c be the number whose hyperbolic logarithm is 1: then $c^{2z} = 1+x$: therefore $dx = 2dz \cdot c^{2z}$,

$$x = c^{2z} - 1, \text{ and } \frac{dx}{x} \log. (1+x) = \frac{4zc^{2z} \cdot dz}{c^{2z} - 1} =$$

$$\frac{4zc^{2z} \cdot dz}{c^2 - c^{-2}} = 2zdz + 2dz \times \frac{z(c^2 + c^{-2})}{c^2 - c^{-2}}. \text{ Let } \frac{z(c^2 + c^{-2})}{c^2 - c^{-2}} =$$

$$1 + \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6} \&c.$$

$$\frac{z^2}{1.2.3} + \frac{z^4}{1.2.3.4.5} + \frac{z^6}{1.2.3.4.5.6.7} \&c.$$

into a series $1 + A^{(0)} z^2 - A^{(4)} z^4 + A^{(6)} z^6 - \&c.$ then by substituting, and taking the fluents, we get

$$f(x) = z^2 + 2 \times \left\{ z + A^{(0)} \cdot \frac{z^3}{3} - A^{(4)} \cdot \frac{z^5}{5} + A^{(6)} \cdot \frac{z^7}{7} - \&c. \right\}.$$

XXVII. QUESTION 177, by MARLOVIENSIS.

It is required to shew a simple practical method (capable of demonstration), of putting a globe or sphere into perspective; having given the magnitude and situation of the sphere, the height of the eye above the ground plane, and its distance from the plane

plane of projection, which is, as usual, supposed to be perpendicular to the horizon, also to shew what situation the globe must have, that the contour of its perspective may be any proposed conic section ?

SOLUTION, *by Mr. LOWRY.*

The rays proceeding from that part of the globe which is visible to the eye, will evidently form a right cone whose vertex is at the eye, and its base a circle made by the contact of the sphere with the visual rays that form the conic surface; and the contour of the perspective representation of the globe, on the picture, will be the figure made by the intersection of the plane of the picture with the conic surface; and will be one or other of the conic sections according to the relative positions of the cone and plane. Conceive a plane to pass through the eye in a direction parallel to the plane of projection, or plane of the picture: this plane is generally called the directing plane: then, when the globe is placed either on the contrary side of the picture to the eye, or between the picture and the directing plane, but so as not to touch that plane, the cone of rays will be intersected on all sides by the plane of the picture, and the figure will be an ellipse, except in that particular case when the base of the cone is parallel to the plane of the picture, and then the figure will be a circle. Again, when the globe touches the directing plane on the side next to the picture, a side of the cone will be in the directing plane, and consequently will be parallel to the plane of the picture, and the section will be a parabola: but when the globe is placed so as to be cut by the directing plane, it is evident that the part of the globe on the side next to the picture will only be visible to the eye, and the figure will be a hyperbola. Hence, we see, that the contour of the perspective of the globe, will be an ellipse, when the globe is placed without the directing plane; a parabola when it touches the directing plane, on the side next to the picture; and a hyperbola when the globe is cut by the directing plane.

Let E (fig. 93, 94, pl. 4.) be the position of the eye, *ABab* the plane of projection, or, as it is usually called, the plane of the picture, and *H* the centre of the sphere. Also, let *EC* be drawn perpendicular to the plane *ABab*, and conceive a plane to pass through the eye, the centre of the picture, and the centre of the sphere; or through the points *E, C, H*, meeting the plane of the picture in the line *CI*, and the horizontal plane passing through *H* in the line *HI*. Then, since the relative positions of the globe, the eye, and the plane of the picture are given; the distances *IH, IC*, and *EC* are given. Let the rays *EG, EF* touch the circle *FGW*, which is the section of the sphere by the plane *ECH*, in *G* and *F*, and meet *CI* in *g* and *f*, and join *GF*; then *GF* is evidently the

diameter of the base of the visual cone, and gf its perspective representation on the plane of the picture. And since gf passes through the axis of the cone, and also through the point C , where a perpendicular from the vertex of the cone meets the intersecting plane, it is easy to prove that when the cone is right, gf is one of the axes of the section, namely, the greater axis when the figure is an ellipse, and the greater, or less axis, according as the cone is acute or obtuse-angled, when the figure of the section is a hyperbola. It is also evident that the other axis of the section will be parallel to the base of the cone.

Bisect gf in S the centre of the ellipse or hyperbola, and draw MSN parallel to GF , meeting EG in N and EF in M ; and let PSQ , drawn perpendicular to gf , be the other axis; then the points P, Q, N, M being in the conical surface, and in a plane parallel to the base, they are in the circumference of a circle, and $MS \cdot SN = PS \cdot SQ =$ to the square of the semi-axis SP or SQ . From hence we derive the following practical construction: take a line hi = to the distance HI , and with a distance equal to the radius of the sphere, and centre h , describe a circle. Make ic perpendicular to ih and equal to the distance IC ; and make ce perpendicular to ic and = to CE , the distance of the eye from the plane of the picture; draw the tangents eG', eF' meeting ic in g' and f' ; then it is evident that the distances if', eg' are respectively equal to the distances If, Cg , and therefore the axis fg may be determined by making $If = if'$, and $Cg = to cg'$. Let $f'g'$ be bisected in s , and through s draw mn parallel to $F'G'$; then ms, sn are respectively equal to MS, SN , and therefore the rectangle $ms.sn$ is = to the square of the semi-axis PS or SQ , hence if PS or SQ be taken each equal to a mean proportional between ms and sn , we have the two axes of the section and the perspective curve may be described by known methods.

The figure is drawn only for the case when the contour of the perspective of the globe is an ellipse, but the above construction is equally applicable when the contour is an hyperbola, as will be very evident to any one who will take the trouble to draw the figure.

Instead of finding the axis PQ , we might have found an ordinate OL , passing through the point O where the axis fg meets EH the axis of the cone. For if GF meet EH in K , it is evident (since OL is parallel to the base of the cone) that $EO : LO :: EK : \text{the radius of the base of the cone}$; and since EO, EK (or eo, eK') and the radius KF are given; it is evident that LO is given, and it may be found by drawing ol parallel to $G'F'$; for then OL is = ol . This method of construction is immediately applicable when the contour of the perspective of the globe is a parabola.

XXVIII. QUESTION 178, by MECHANICUS.

If two given bodies, A and B, connected by a string, be supported on an inclined plane, by means of the string passing through a ring at A, and any given force be impressed on the body A, in a horizontal direction; it is required to determine the nature of the curve described by the body, and the circumstances of the motion?

SOLUTION, by MECHANICUS, the Proposer.

Let CDEF (fig. 186, pl. 4.) be the inclined plane, n the sine of its inclination to the horizon, R the position of the ring, or the point where the body A is placed at the commencement of the motion; A and B the position of the bodies at the end of any indefinite time t , and RAa a portion of the curve described by the body A. Also let QR, parallel to the horizon, be the direction of the impulse which communicates motion to the bodies, and draw AQ perpendicular to RQ, or parallel to the string RB.

Let Aa be an indefinitely small portion of the curve, draw am perpendicular to QAS, and join AR, aR; with the distance AR and centre R describe the small arc An and take Bb = an.

Put l = the length of the string,

$g = 32\frac{1}{2}$ feet, the measure of the accelerative force of gravity,

$x = AR$, and then BR is $= l - x$,

$v =$ the velocity in the curve at A,

$u =$ the velocity in the direction AR, or the velocity with which the bodies approach to or recede from the ring,

$y = AQ$, $z = RQ$, $\dot{s} = Aa$, $\dot{x} = an = Bb$, $\dot{y} = am$.

The impulse given to the body A, when at R, causes it to recede from the ring and draw up the body B on the inclined plane in the direction BR: the motive force of A in the direction AS is $= ng \times A$, which being resolved into the direction RA is $= ng \times A \frac{\dot{y}}{\dot{x}}$; and since B acts in opposition to this force, its effect in the direction of the string must be subtracted from $ng \times A \frac{\dot{y}}{\dot{x}}$ to obtain the motive force, which, upon the whole, urges A or B towards the ring; but the motive force of B, or the force that urges it to descend in the direction BR is $= ng \times B$, therefore $ng \times (A \frac{\dot{y}}{\dot{x}} - B) = ng \times (\frac{A\dot{y} - B\dot{x}}{\dot{x}})$ is the motive force in the direction of the string. And this force resolved into the direction Aa is $= ng \times (\frac{A\dot{y}}{\dot{x}} - B) \times \frac{\dot{x}}{\dot{s}} = ng \times (\frac{A\dot{y} - B\dot{x}}{\dot{s}})$ the motive force in the direction Aa of the

curve. These forces divided by the inertia of the mass moved, are equal to the respective accelerative forces in the directions AR, Aa.

The inertia of B is evidently equal to its own mass B; but since A and B move with different velocities, or are differently accelerated, the inertia of A cannot be equal to its own mass but to some other quantity. Let this quantity be M, that is, let M be a mass such, that when placed at B, and the mass A removed, it may resist the communication of motion in the same manner as the mass A when it moves in its proper path.

Now the motive force of A, in the direction Aa, being $ng \times \left(\frac{A\dot{y} - B\dot{x}}{\dot{s}}\right)$, the accelerative force in that direction is $ng \times \left(\frac{A\ddot{y} - B\ddot{x}}{\dot{s}A}\right)$ if A only be considered as the mass moved; and the motive force of M (or B), in the direction BR, being $ng \times \left(\frac{A\dot{y} - B\dot{x}}{\dot{x}}\right)$, the accelerative force in that direction is $ng \times \left(\frac{A\ddot{y} - B\ddot{x}}{M\dot{x}}\right)$ if M only be considered as the mass moved:

But A describes the space $\dot{s}$ with the velocity v , in the same time that M describes the space $\dot{x}$ with the velocity u , and the accelerative force being equal to the fluxion of the velocity divided by the fluxion of the time, the accelerative forces of A and M will be as the fluxions of the velocities, that is,

$ng \times \left(\frac{A\ddot{y} - B\ddot{x}}{A\dot{s}}\right) : ng \times \left(\frac{A\ddot{y} - B\ddot{x}}{M\dot{x}}\right) :: \dot{v} : \dot{u}$; therefore

$M = \frac{A\dot{s}\dot{v}}{\dot{x}\dot{u}}$; and $B + \frac{A\dot{s}\dot{v}}{\dot{x}\dot{u}}$ = the whole inertia of the mass.

Consequently, $ng \times \left(\frac{A\ddot{y} - B\ddot{x}}{\dot{x}}\right) \div \left(B + \frac{A\dot{s}\dot{v}}{\dot{x}\dot{u}}\right) = ng \times \frac{(A\ddot{y} - B\ddot{x}) \dot{u}}{B\dot{u}\dot{x} + A\dot{v}\dot{s}}$ is the accelerative force in the direction AR or

Bb, which, by dynamics, is = to $\frac{u\dot{u}}{\dot{x}}$; therefore $ng(A\ddot{y} - B\ddot{x})$

$= Bu\dot{u} + \frac{A\dot{s}u\dot{v}}{\dot{x}}$, or since $\frac{\dot{s}}{\dot{x}} = \frac{v}{u}$

$$ng(A\ddot{y} - B\ddot{x}) = Bu\dot{u} + Av\dot{v} \dots \dots \dots (1)$$

and taking the fluents

$$2ng(A\dot{y} - B\dot{x}) = Bu^2 + Av^2 \dots \dots \dots (2).$$

Now, besides the force which acts in the direction of the string, there is another force which disturbs the motion of the bodies by acting in a direction perpendicular to the radius vector RA, for the motive force of A in the direction QSA is $ng \times A$, which

which being resolved into a force perpendicular to RA is $ng \times A \times \sin. QAR = ng \times A \times \frac{z}{x} = A \times f$ (supposing $ng \times \frac{z}{x} = f$).

To compute the effect of this force on the motion of the bodies; let w be the velocity of A, in a direction perpendicular to the radius vector AR, and let ρ be the value of the perpendicular drawn from R to the tangent to the curve at A: then the velocity in the direction of the tangent, or element Aa of the curve, will be $\frac{w}{\sin. PAR} = \frac{wx}{\rho}$. This velocity would continue invariable, if the forces were to cease to act on the body A,

and its fluxion $\frac{w\dot{x} + \dot{w}x}{\rho}$ would then be $= 0$; or $\dot{w}$ would be $= -\frac{w\dot{x}}{x}$; but when the forces continue to act, the fluxion

of the velocity is $\dot{w}$, therefore $\dot{w} + \frac{w\dot{x}}{x}$ (or the excess of the fluxion of the velocity when the forces act, above the fluxion of the velocity when the forces cease to act,) will be equal to the fluxion of the velocity generated or destroyed by the action of the motive force $A \times f$ only. But the fluxion of the velocity is $=$ to the element of the time i (or $\frac{\dot{x}}{u}$) multiplied by the accelerative force, that is,

$$\dot{w} + \frac{w\dot{x}}{x} = \frac{\dot{x}}{u} \times f, \text{ or } f = \left(\dot{w} + \frac{w\dot{x}}{x} \right) \times \frac{u}{\dot{x}}.$$

But it is evident that w is $= \sqrt{(v^2 - u^2)}$, therefore $\dot{w} = \frac{v\dot{v} - u\dot{u}}{\sqrt{(v^2 - u^2)}}$; and putting these values in the expression for f , we have

$$f = \left(\frac{v\dot{v} - u\dot{u}}{\sqrt{(v^2 - u^2)}} + \sqrt{(v^2 - u^2)} \times \frac{\dot{x}}{x} \right) \times \frac{u}{\dot{x}},$$

$$\text{or, } ngz (=fx) = \frac{(v\dot{v} - u\dot{u})xu}{\dot{x}\sqrt{(v^2 - u^2)}} + u\sqrt{(v^2 - u^2)} \dots\dots (3).$$

From the equations (1) (2) (3), and recollecting that $\dot{s} = \sqrt{(\dot{y}^2 + \dot{z}^2)}$, $\frac{\dot{s}}{\dot{x}} = \frac{v}{u}$, and $z^2 = x^2 - y^2$, we may determine the circumstances of the motion, and also the relation between y and z , the rectangular co-ordinates of the curve; but the expressions are too complicated to admit of any elegant conclusion.

XXIX.

XXIX. QUESTION 179, by Mr. WALLACE, R. M. College.

Suppose two particles of matter p and q , to be connected by a string pAq , and to rest in equilibrio upon two inclined planes AB , AC , the string being supposed to pass over the common section of the planes at the point A ; then, if a circle be described through the points p , A , q , and the particles without changing their position are disengaged from the string pAq , and connected by another paq , which passes over a peg fastened at any point a in the circumference of the circle, the particles p and q will still rest in equilibrio upon the planes. Required the demonstration?

SOLUTION, by Mr. THOMAS BAZLEY, Bolton.

Fig. 87, pl. 4. Draw the horizontal line pQ cutting AC in Q ; then, by the property of the inclined plane (when the string passes over the common section of the planes) $\text{rad. } 1 : \sin. ApQ :: p : p \times \sin. ApQ = \text{the force sustaining } p$, and in like manner $q \times \sin. AQP = \text{the force sustaining } q$; whence by the question $\sin. ApQ \times p = \sin. AQP \times q$.

Now when the string passes over a , we shall have, (by cor. 1. prop. 15, Emerson's Mechanics, 8vo.), $\text{cos. } apA : \sin. ApQ :: p : (\sin. ApQ \div \text{cos. } apA) \times p = \text{the force of } p \text{ in the direction } ap$; and for the same reason, $(\sin. AQP \div \text{cos. } Aqa) \times q = \text{the force of } q \text{ in the direction } aq$. But it was shewn that $\sin. ApQ \times p = \sin. AQP \times q$, and, by the property of the circle, the angle ApA is = to the angle Aqa ; therefore $\frac{\sin. ApQ}{\text{cos. } ApA} \times p = \frac{\sin. AQP}{\text{cos. } Aqa} \times q$; consequently p and q sustain each other at a .

And thus the question was answered by Messrs. Cunliffe, Nicholson, and Simpson.

XXX. PRIZE QUESTION 180, by GEOMETRICUS.

If the opposite sides of a hexagon, inscribed in a circle, be produced till they meet, the three points of intersection will be in the same straight line. Required the demonstration?

FIRST SOLUTION, by the late Earl STANHOPE.

(From the Appendix to Schawb's edition of Euclid's Data.)

Fig. 88, pl. 4. Let the opposite sides of the hexagon $AFEDCB$, when produced, meet in the points G , H , I .

Draw BL parallel to EI , and FM parallel to CG ; and through F , B , draw LP , MN ; join PH , NH , and draw FC and BE .

Because, by construction, BL is parallel to ED , the arc $BCD = LFE$, and the angle $BCD = LFE$, and their supplements or the angles PCH , PFH are equal; and since the angle $FQP = CQH$, the remaining angles of the triangles FQP , CQH are equal, or the angle $CHQ = FPQ$. Therefore the quadrilateral $FCHP$ may be circumscribed by a circle. And in the same manner it is proved that a circle will circumscribe the quadrilateral $BEHN$.

Hence

Hence the angle $\text{CFH} = \text{CPH}$: but the angles CFH , CDN are equal ; therefore the angle $\text{CPH} = \text{CDN}$. In like manner it is proved that the angle $\text{EDP} = \text{ENH}$, therefore EI is parallel to PH , and DG to HN , and consequently DNHP is a parallelogram ; therefore the angle $\text{QPH} = \text{RNH}$.

Again, the angle $\text{AFM} = \text{AGC}$, and $\text{AFM} = \text{ABM}$, therefore $\text{ABM} = \text{AGC}$. But because the angle $\text{ABM} = \text{IBN}$, the angle $\text{IBN} = \text{AGC}$. And in the same manner it is proved that the angle $\text{GFP} = \text{BIN}$. Therefore the triangles FGP , BIN are equiangular ; consequently the supplement of the angle BNI or the angle BNR is equal to FPQ the supplement of FPG . And since the angle $\text{QPH} = \text{RNH}$, therefore $\text{FPQ} + \text{QPH}$ or $\text{FPH} = \text{BNR} + \text{RNH}$. But $\text{HBN} = \text{HEN}$, and $\text{HEN} = \text{FHP}$, therefore the triangles BNH , FPH are similar. And because the triangles FGP , BIN are also similar, we have

$\text{GP} : \text{PF} :: \text{BN} : \text{NI}$; and $\text{PH} : \text{PF} :: \text{BN} : \text{HN}$;

therefore (*ex æq. perturb.*) $\text{GP} : \text{PH} :: \text{HN} : \text{NI}$;

whence (by comp.) $\text{GP} : \text{PH} :: \text{GP} + \text{HN}$ (or GD) : $\text{PH} + \text{NI}$ (or DI) ;
therefore GHI is a straight line.

SECOND SOLUTION, by Mr. IVORY.

Fig. 89, pl. 4. Let the opposite sides of the hexagon meet in the points G , H , I .

Join two of the opposite angles of the hexagon, as A and D ; about the triangle ADI describe a circle, and let that circle meet GA and GD (or these lines produced) in K and L ; and join FC , KI , LI , KL . Because the quadrilateral ADIK is in a circle, therefore the angle $\text{AKI} = \text{ADE}$: and because the quadrilateral FADE is in a circle, therefore the angle $\text{ADE} = \text{GFE}$: therefore the angle $\text{AKI} = \text{GFE}$, and KI is parallel to FH . In like manner, on account of the circles, the angle $\text{AIL} = \text{ADL} = \text{CBI}$; and LI is parallel to HB . And, again, the angle $\text{KLD} = \text{DAF} = \text{FCD}$; and KL is parallel to FC . Because FC is parallel to KL , therefore

$\text{GF} : \text{FC} :: \text{GK} : \text{KL}$.

And, because the sides of the triangle HFC have been shewn to be parallel to the sides of the triangle IKL , each to each, therefore

$\text{FC} : \text{FH} :: \text{KL} : \text{KI}$.

Therefore, *ex æquo*, $\text{GF} : \text{FH} :: \text{GK} : \text{KI}$.

Therefore G , H , I are in one straight line.

THIRD SOLUTION, by Mr. LOWRY.

Let ABCDEF (fig. 90, pl. 4.) be the inscribed hexagon ; G the intersection of the opposite sides BC , EF ; H the intersection of the opposite sides AB , DE ; and I the intersection of the opposite sides CD , AF ; then the points G , H , I are in a straight line.

Draw HK parallel to BC meeting CD produced in K , and GL parallel to AB meeting IF in L . Join AK , LC , AD and CF . Then, because HK is parallel to BC ; the angle BCI (or BAD

BAD, Euc. iii. 22.) is $\equiv$ to the angle HKD; therefore the points A, D, K, H are in the circumference of the same circle, and the angle ADH $\equiv$ to the angle AKH.

Again, because GL is parallel to BA, the angle GLF is $\equiv$ to the angle HAF or (*ibid.*) to the angle BCF; therefore the points G, L, C, F are in the same circle, and the angle GCL $\equiv$ the angle GFL $\equiv$ the angle ADH or AKH; therefore LC is parallel to AK, and consequently the triangle ILC is equiangular to the triangle IAK, and the triangle GLC equiangular to the triangle HAK; therefore $LG : AH :: LC : AK :: IL : IA$; and since the points I, L, A are in a straight line, it is evident that the points I, G, H are also in a straight line.

The proposition may be enunciated more generally as follows:

Let two chords AB, DE (fig. 91, pl. 4.) be inscribed in a circle, intersecting at H (either within or without the circle), and from two of the terminations B, D (or B, E) let two straight lines be inflected to any point C in the circle, and from the other terminations A, E (or A, D) let two other straight lines be inflected to any point F in the circle, meeting the former inflected lines in G and I; then the points G, H, I will be in a straight line.

If any two of the points in the circle coincide, a tangent may be substituted for the inflected line, and the proposition will still be true, and the preceding demonstration will hold good.

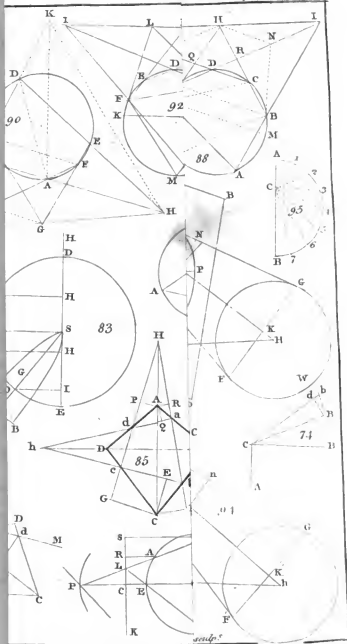
The proposition thus generalized, is also true in any of the conic sections, as was, I believe, first discovered by Maclaurin. It is also true when the figure is inscribed in a small circle on the surface of a sphere; arches of great circles being substituted in place of straight lines.

FOURTH SOLUTION, by Mr. RICH. NICHOLSON, *Liverpool*.

Let ABCDEFA (fig. 92, pl. 4.) be any irregular hexagon inscribed in a circle; produce the sides AB, ED and BC, EF to intersect in G and H; draw GH, which produce to meet CD in I, and join IF, FA: then, if the proposition be true, FA is a continuation of IF. To demonstrate this, draw BK parallel to GI, and KL through F to meet GI in L; join F, C and L, C: then, the angles GHB, KBH and LFC being equal, the points F, C, H, L are in a circle; therefore the angles HLC, HFC and GDC are equal and the points G, C, D, L are in a circle; wherefore the angles KFA, KBA and LGB are equal, and the points A, G, L, F are also in a circle. Now, if FA be not a continuation of IF let FM be a continuation of it meeting the circle in M; then $IG \times IL = IC \times ID = IM \times IF$, therefore the points G, F, L, M are in the same circle as G, L, F, A; therefore, joining M, B; M, G, the angle MGA is equal to the angle MFA, that is, to the angle MBA, which is absurd. Wherefore FA is a continuation of IF, and the proposition of Geometricus true.

Mr. IVORY is requested to send to Mr. GLENDINNING's for the Medal for solving the Prize Question.

Plate IV. Fig. 73 to 95.



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ARTICLE III.

Solutions to Questions proposed in Number VI.

I. QUESTION 181, by YANTO.

Bacchus caught Silenus asleep by the side of a full cask, he seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that, Silenus awakes, and drinks what Bacchus had left. Had they drank both together, it would have been emptied two hours sooner, and Bacchus would have drank only half of what he left for Silenus. Required the time in which they would empty the cask separately?

SOLUTION, by YANTO, the Proposer.

Let x denote the number of hours in which Bacchus alone would empty the cask, and y the number of hours in which Silenus would do the same. Then if the content of the cask be

denoted by unity, $\frac{1}{x}$ and $\frac{1}{y}$ will express the respective quanti-

ties drank by Bacchus and Silenus in an hour; and by the question, $\frac{2}{3}y$ is the time Bacchus continued drinking while Silenus was asleep; therefore, by the question, we shall also have $x :$

$\frac{2y}{3} :: 1 : \frac{2y}{3x} =$ the quantity drank by Bacchus whilst Silenus

slept; and $1 - \frac{2y}{3x} = \frac{3x - 2y}{3x} =$ the quantity which Sile-

nus drank: and therefore, $\frac{1}{y} : 1 :: \frac{3x - 2y}{3x} : \frac{y(3x - 2y)}{3x}$

$=$ the time Silenus was drinking: consequently, $\frac{2}{3}y + \frac{y(3x - 2y)}{3x} = \frac{5xy - 2y^2}{3x}$ will express the whole time of

drinking the cask. Again, $\frac{1}{x} + \frac{1}{y} : 1 :: 1 : \frac{xy}{x + y} =$ the

time in which both together might empty the cask ; and therefore by the question,

$$\frac{5xy - 2y^2}{3x} = \frac{xy}{x+y} + 2.$$

Also, $\frac{1}{x} + \frac{1}{y} : 1 :: \frac{1}{x} : \frac{y}{x+y} =$ Bacchus's share, had they drank both together ; which, by the question, must be equal to half of what was left for Silenus ; that is,

$$\frac{3x - 2y}{3x} = \frac{2y}{x+y}, \text{ or } 3x^2 - 5xy = 2y^2 :$$

Hence, $x = 2y$, or $-\frac{1}{3}y$; but as the question does not admit of x or y being negative, the only value of any use, is $x = 2y$, by means whereof, exterminate x out of the equation $\frac{5xy - 2y^2}{3x} = \frac{xy}{x+y} + 2$, and it becomes $\frac{4y}{3} = \frac{2y}{3} + 2$: whence $y = 3$ and $x = 2y = 6$.

And thus the Question was answered by Messrs. Amicus, Cunliffe, O'Riordan, Toplis, and Vinofus.

II. QUESTION 182, by AMICUS.

Two given beams, or rods, loaded with weights, being placed on a horizontal plane, so that their lower extremities may be opposed to each other, and the other extremities be supported by two parallel vertical planes ; it is required to find their position when they are in equilibrio, and the pressure exerted against each plane ?

FIRST SOLUTION, by Mr. CUNLIFFE, R. M. College.

Let AB, (fig. 96, pl. 5:) represent the horizontal plane upon which the two beams or rods are placed ; AM and BP the two vertical planes ; NM and NP the two rods or beams opposed to each other at their lower ends N, and the other ends leaning against the vertical planes AM and BP.

Let G, g represent the respective centres of gravity of the beams NM and NP : draw MR, PS parallel to AB and RGH, SgL perpendicular thereto, and join RN, SN. Let w denote the weight of the beam NM, and w' the weight of the beam NP.

Then by the corollary to Prop. 63, Emerson's Mechanics,
RH

$RH : HN :: w : (HN \div RH) \times w =$ the force exerted by the beam NM in the direction AN : and for the same reason, $(LN \div SL) \times w' =$ the force exerted by the beam NP in the direction BN : and these opposite forces when the beams are in equilibrio, must counteract or be equal to each other; that is

$$(HN \div RH) \times w = (LN \div SL) \times w',$$

or $HN \times SL \times w = LN \times RH \times w'.$

And by reason of the parallels $HG, AM; Lg, BP,$

$$NG : NM :: NH : NA = NM \times NH \div NG;$$

$$Ng : NP :: NL : NB = NP \times NL \div Ng.$$

Put $NG = a, GM = b, Ng = c, gP = d, NH = x, NL = y,$ and $AB = s.$ Then $GH = \sqrt{(a^2 - x^2)},$ and $gL = \sqrt{(c^2 - y^2)},$ and from what has been deduced, $NA = (a + b)x \div a$ and $NB = (c + d)y \div c.$

Again, because of the parallels $HG, AM; Lg, BP,$
 $NG : NM :: HG : AM = (a + b) \sqrt{(a^2 - x^2)} \div a = RH,$
 $Ng : NP :: Lg : BP = (c + d) \sqrt{(c^2 - y^2)} \div c = SL,$ by means of which, the equation $HN \times SL \times w = LN \times RH \times w'$ becomes

$$wx(c + d) \sqrt{(c^2 - y^2)} \div c = w'y(a + b) \sqrt{(a^2 - x^2)} \div a,$$

also, $NA + NB = (a + b)x \div a + (c + d)y \div c = s.$

From the two preceding equations, x and y may be found, which will determine the position of the beams when in equilibrio. And the pressure of each beam against its respective vertical plane, is equal to that exerted by its lower end in an horizontal direction, (cor. prop. 63, Emerson's Mechanics), and therefore the pressure against each of the vertical planes when the beams are in equilibrio, is expressed by $(HN \div RH) \times w.$ The pressure against the horizontal plane is obviously equal to the weight of both beams or $w + w'.$

SECOND SOLUTION, by the Rev. I. TOPPIS.

Let $AG, MP,$ (fig. 97, pl. 5.) represent the two vertical planes, GM the horizontal one, AH, PH the two beams, D and Q their centres of gravity: let DF and $QS,$ perpendicular to the horizon, represent the respective weights of the loaded beams. From D draw DE perpendicular, and FE parallel to AH : then DF may be decomposed into the forces $DE, EF.$ In like manner, let QR perpendicular, and RS parallel to $PH,$ represent the force $QS,$ decomposed into these directions. Let AC perpendicular to $AG,$ represent the re-action of the plane AG upon the beam $AH,$ which re-action may be decomposed into two others, AB perpendicular, and BC parallel to $AH.$ In like manner, PN may be decomposed into PT perpendicular, and TN parallel to $PH.$ In the direction of the beam $AH,$ let

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Hc

Hc be taken equal to $BC + EF$, and decomposed into cd and dH ; then Hc represents the force which acts upon the beam in the direction AH . Let Ha be the force with which the beam AH presses upon the plane GM , in a direction perpendicular to itself, which may be decomposed into the forces Hb parallel, and ba perpendicular to GH . Let $He = TN + RS$ represent the force of the beam PH , in the direction PH , which may be decomposed into Hf parallel, and fc perpendicular to GM ; also the pressure Hg , of the end of the beam PH , may be decomposed into the forces Hh parallel, and hg perpendicular to GM .

If $AB = HD \times DE \div AH$, $cH = BC + EF$, $Ha = AD \times DE \div AH$, $PT = QR \times QH \div PH$, $cH = TN + RS$, $Hg = QR \times QP \div PH$, and $Hd - Hb = Hf - Hh$, it is evident, from the known principles of Mechanics, that the beams will be in equilibrio.

To calculate this position, we may easily perceive that the right-angled triangles AGH , ABC , DEF , Hcd and Hab have the angles GAH , BAC , EFD , Hcd and Hab equal, therefore the triangles are similar. Also the right-angled triangles PTN , PHM , QRS , Hcf and Hgh are similar, for the angles NPT , HPM , QSR , fcH , and gHh are evidently equal.

Put $\angle GHA = x$, $\angle PHM = y$, $DF = a$, the weight of the beam AH ; $DA = b$, $DH = c$, $AH = b + c = d$, $QS = m$, the weight of the beam PH ; $QH = h$, $QP = k$, $PH = h + k = f$, $GM = g$; then $DE = a \cos. x$, and $EF = a \sin. x$,

$$AB = HD \times DE \div AH = \frac{ac}{d} \cos. x, \text{ and } BC = \frac{AB}{\tan. x} =$$

$$\frac{ac}{d} \cdot \frac{\cos. x}{\tan. x}; \text{ therefore } Hc = BC + EF = \frac{ac}{d} \cdot \frac{\cos. x}{\tan. x} + a \times$$

$$\sin. x, \text{ and } Hd = Hc \cdot \cos. x = \frac{ac}{d} \cdot \frac{\cos.^2 x}{\tan. x} + a \sin. x \cos. x.$$

$$\text{Also } Ha = AD \times DE \div AH = \frac{ab}{d} \cos. x, \text{ and } HB = Ha$$

$$\times \sin. x = \frac{ab}{d} \cos. x \sin. x; \text{ therefore } Hd - Hb = \frac{ac}{d} \cdot \frac{\cos.^2 x}{\tan. x}$$

$$+ \left(a - \frac{ab}{d}\right) \cos. x \sin. x; \text{ but } \left(a - \frac{ab}{d}\right) \cos. x \sin. x =$$

$$\frac{ac}{d} \cdot \cos. x \sin. x = \frac{ac}{d} \cdot \frac{\sin.^2 x}{\tan. x}; \text{ hence } Hd - Hb = \frac{ac}{d}$$

$$\left(\frac{\cos.^2 x + \sin.^2 x}{\tan. x}\right) = \frac{ac}{d \tan. x}. \text{ By calculating the forces}$$

$$Hf, Hh, \text{ in a similar way, we find } Hf - Hh = \frac{mh}{f \tan. y}, \text{ and}$$

since

since $Hd - Hb$ is $= Hf - Hh$, we have

$$\frac{ac}{d \tan. x} = \frac{mh}{f \tan. y} \dots \dots \dots (1)$$

Also, $GH = AH \cos. x = d \cos. x$, and $HM = PH \cos. y = f \cos. y$; therefore

$$d \cos. x + f \cos. y = g \dots \dots \dots (2)$$

And from the equations 1 and 2, we may easily determine the angles x and y , and then the position of the beams is known.

To find the pressure against the planes, we have $AC = \frac{AB}{\sin. x} = \frac{ac \cos. x}{d \sin. x} = \frac{ac}{d \tan. x}$, the pressure against the plane GA ;

and $PN = \frac{PT}{\sin. y} = \frac{mh}{f \tan. y}$, the pressure against the plane MP .

Again, $cd = cH \sin. x = \frac{ac}{d} \cdot \cos. x \frac{\sin. x}{\tan. x} + a \sin.^2 x$, and

$ab = aH \cos. x = \frac{ab}{d} \cos.^2 x$, and $cd + ab = \frac{ac}{d} \times$

$\frac{\cos. x \sin. x}{\tan. x} + a \sin.^2 x + \frac{ab}{d} \cos.^2 x$; but $\frac{\cos. x \sin. x}{\tan. x}$

$= \cos.^2 x$; therefore $cd + ab = a \left(\frac{c + b}{d} \cos.^2 x + \right.$

$\left. \frac{d}{d} \sin.^2 x \right) = a$, the pressure of the beam AH against the

plane GM ; and by a similar calculation, we find $ef + gh = m$, the pressure of the beam PH against the plane GM , or the whole pressure of the beams against the horizontal plane, is equal to the sum of their weights.

THIRD SOLUTION, by AMICUS, the Proposer.

Let GA and MP , (fig. 98, pl. 5.) be the vertical planes, AH , PH the given beams placed in the required position, D the centre of gravity of the beam AH , and W its weight; and Q the centre of gravity of the beam PH , and w its weight. Join DQ , and make $DL : LQ :: W : w$; then L is the common centre of gravity of the beams: and when an equilibrium takes place, this centre must be at the lowest point, or at the least distance from the horizontal plane GM ; that is, if LX be perpendicular to GM , then LX must be a minimum. Draw DI , QK perpendicular to GM , and put $a = DH$, $b = AH$, $m = QH$, $n = PH$

PH, $p = GM$, $x = \angle GHA$, and $y = \angle PHM$; then $DI = a \sin. x$, and $QK = m \sin. y$. Draw QV parallel to GM , meeting LX in Y , and DI in V ; then $LX = QK + LY$, and $DV = DI - QK$, and $LQ : QD :: LY : DV$, or $LY =$

$VD \times \frac{LQ}{QD} = (ID - QK) \frac{w}{w + W} = \frac{w}{w + W} \times (a \sin. x - m \sin. y)$; therefore $QK + LY = m \sin. y + \frac{w}{w + W} (a \sin. x - m \sin. y) = \frac{w}{w + W} (a \sin. x) + \frac{Wm}{w + W} \sin. y = a \text{ minimum.}$ Take the fluxion, then

$$\frac{wa}{w + W} \dot{x} \cos. x + \frac{Wm}{w + W} \dot{y} \cos. y = 0,$$

$$\text{or } \dot{y} = - \frac{wa \dot{x} \cos. x}{Wm \cos. y}.$$

Again, $GH = AH \cos. x = b \cos. x$, $HM = n \cos. y$, and $b \cos. x + n \cos. y = p$, a constant quantity, therefore its fluxion, or $-b \dot{x} \sin. x - n \dot{y} \sin. y = 0$, and $\dot{y} = - \frac{b \dot{x} \sin. x}{n \sin. y}$:

Hence $\frac{wa \dot{x} \cos. x}{Wm \cos. y} = \frac{b \dot{x} \sin. x}{n \sin. y}$, or $\frac{wa \cos. x}{Wm \cos. y} = \frac{b \sin. x}{n \sin. y}$,

or $\frac{n \sin. y}{Wm \cos. y} = \frac{b \sin. x}{wa \cos. x}$, or $\frac{n}{Wm} \tan. y = \frac{b}{wa} \tan. x$, and

from this equation, and the equation $b \cos. x + n \cos. y = p$, we may determine x and y .

A solution was also received from Dr. O'Riordan.

III. QUESTION 183, by QUIDAM.

Given all the four sides, and the difference of the parts into which one of the angles is divided by a diagonal; to construct the trapezium and point out the limits.

SOLUTION, by Dr. MICHAEL STUART O'RIORDAN, Carlow, Ireland.

Suppose it done, and let $ABCD$, (fig. 98, pl. 5,) be the trapezium required, in which the four sides and the difference of the angles DCA , BCA , (made by one of the diagonals AC) are

given

given. Let the triangle AOC be made equal to the triangle ADC, and join BO; that is, let OC be made $\equiv$ to CD, and AO $\equiv$ to AD, and then the angle ACO will be equal to the angle ACD, and the angle BCO will be equal to the difference of the angles DCA, BCA: hence the following

CONSTRUCTION. With the given sides CB, CO (CD), and given difference of the angles make the triangle CBO, and with the given distances OA (AD), BA, and centres O, B, describe arcs to intersect at A; make the triangle ACD $\equiv$ to AOC, and it is done.

It is evident, that BO must not be less than the difference, nor greater than the sum of OA, BA; and when BO is equal to the difference of AO, BA the angle BCO, or the difference of the angles DCA, BCA is a minimum.

Messrs. Amicus and Cunliffe, answered it.

IV. QUESTION 184, by Mr. JOHNSON, Birmingham.

A given quantity of gold is to be made into a cup in the form of a segment of a sphere, and n tenths of an inch in thickness: what are the dimensions of the cup when it is made so as to contain the greatest quantity possible?

SOLUTION, by Mr. CUNLIFFE.

Let r denote the radius of the interior sphere, or that which constitutes the hollow part of the cup, and let x denote the depth thereof, or the versed sine of the segment: then will $r + n$ denote the radius of the exterior sphere, and $x + n$ will denote the versed sine of its segment. And by a well known theorem, $p \times (rx^2 - \frac{1}{3}x^3)$, will express the content of the hollow part, where $p = 3.1416$; and $p \times \left\{ (r+n)(n+x)^2 - \frac{1}{3} \times (n+x)^3 \right\}$ will express the content of the segment of the exterior sphere. Also the difference of the foregoing expressions, viz. $p \times \left\{ (r+n)(n+x)^2 - rx^2 \right\} - \frac{1}{3}p \times (n+x)^3 + \frac{1}{3}px^3$ will express the quantity of gold in the cup, which put $\equiv pq^2n$.

Now the content of the hollow part, viz. $p \times (rx^2 - \frac{1}{3}x^3)$ is to be a maximum by the question, or $rx^2 - \frac{1}{3}x^3$ is to be a maximum; putting its fluxion $\equiv 0$, gives $rx^2 + 2rx\dot{x} - x^2\dot{x} = 0$; whence $r = -\dot{x}(2r - x) \div \dot{x}$.

Furthermore, the expression for the quantity of metal being constant,

constant, its fluxion will be $= 0$, or $\dot{r}(n+x)^2 + 2\dot{x}(r+n)(n+x) - \dot{r}x^2 - 2rx\dot{x} - \dot{x}(n+x)^2 + x^2\dot{x} = \dot{r}(n^2 + 2nx) + \dot{x}(n^2 + 2rn) = 0$, or $\dot{r} = -\dot{x} \times \frac{n+2r}{n+2x}$; therefore $\frac{n+2r}{n+2x} = \frac{2r-x}{x}$ which equation, when properly reduced, gives

$x=r$. And hence it appears, that the required segment is an hemisphere. Having now determined that the required segment is an hemisphere, it will be easy to find the radius of the sphere; for, by a well known theorem,

$\frac{2}{3}p \times \{(r+n)^3 - r^3\} = \frac{2}{3}p \times (3r^2n + 3rn^2 + n^3) = pq^2n$, by the question: whence, after proper reduction, completing the square, &c. we shall have $r = \frac{1}{2} \sqrt{(18q^2 - 3n^2)} - \frac{1}{2}n$.

This question was answered by Messrs. O'Riordan and Toplis.

V. QUESTION 185, by QUIDAM.

Let one end of a string of a given length be fastened to a point A in the circumference of a circle, and laid tightly upon the circumference, and brought with the other end T to meet the diameter AB in T: what must be the diameter of the circle that the space BTP included by the right lines TP, TB, and the arc BP may be the greatest possible, TP being a tangent at T?

SOLUTION, by Mr. CUNLIFFE.

Draw the radius CP, (fig. 99, pl. 5,) then $PT \times \frac{1}{2}CP =$ the area of the triangle TPC, and $\text{arc BP} \times \frac{1}{2}CP =$ area of the sector BPC, and their difference, viz. $(PT - \text{arc BP}) \times \frac{1}{2}CP =$ the space BTP included by the right lines TP, TB and the arc BP.

Now let q denote the quadrantal arc of a circle, whose radius is 1; and let a denote the length of the string APT; then $\text{arc AP} + PT = 2q \times CP - \text{arc BP} + PT = a$, whence $PT - \text{arc BP} = a - 2q \times CP$, and hence the space BTP $= (PT - \text{arc BP}) \times \frac{1}{2}CP = (a - 2q \times CP) \times \frac{1}{2}CP = q \times (a \div 2q - CP) \times CP =$ a maximum, or $(a \div 2q - CP) \times CP$ a maximum.

The problem is therefore now evidently reduced to the dividing of a line of a given length into two such parts, that their rectangle may be a maximum, which is known to be the case when the given line

line is bisected; hence we shall have $CP = a \div 4q$, or $AB = 2CP = a \div 2q$. Q. E. I.

From the preceding conclusions, it is easily perceived, that having given the length of the string APT , we may determine the radius CP , so that the area BPT may be equal to a given space, by means of a quadratic equation.

Or which amounts to the same thing, the solution of the following question may be effected by a quadratic equation.

Let a string, $TPAP'$, of a given length, having its ends fastened together, be laid upon the periphery of a circle, and stretched to point T , in the plane of the circle; to determine the radius so that the area PTP' , included by the tangents TP , TP' , and the intercepted arc PP' may be equal to a given space.

An ingenious solution was received from Dr. O'Riordan.

VI. QUESTION 186, by Mr. JOHN WHITLEY.

If an ellipse be circumscribed by a triangle, and lines be drawn from the angles of the triangle to the points of contact, meeting the ellipse again in three points; tangents drawn to the curve at these points will intersect each other in the said lines produced. Also, if the said tangents and the sides of the triangle be produced till they meet, the three points of intersection will be in the same straight line. Required the demonstration?

SOLUTION, by Mr. LOWRY, R. M. College.

The demonstration of this proposition will be greatly simplified by premising the following Lemma.

Let three straight lines, AB , BC , AC , (fig. 100, pl. 5.) touch a conic section in the points P , Q , R , and meet each other in A , B , C ; then, if straight lines AQ , BR , CP be drawn from the points A , B , C , to the points of contact Q , R , P , meeting the curve in a , b , c , they will intersect each other in the same point K ,

Let the line which joins the points Q , R , meet AB in Z , and CP in N , and let K be the intersection of the lines AQ , BR ; then if PC does not pass through K , let PK be drawn to meet QR in some point n different from N ; then (Hamilton's Conic Sections, Prop. IV. B. V.) the line AZ is harmonically divided in the points A , P , B , Z ; therefore the lines AC , PC , BC , are harmonicals with respect to the point Z ; and the lines AQ , Pn , BR , drawn through K , are also harmonicals with respect to the said point Z : wherefore the line ZR , which meets both sets of harmonicals, is harmonically divided in the points Q , N , R , Z , and also in the points Q , n , R , Z , which is impossible, except the points N and n coincide; therefore PC must pass through K .

When KQ is parallel to AB , it is bisected in N , and AB is then bisected in P , and the truth of the lemma is evident.

COR. A tangent to the curve at c , passes through the point Z ; and the points a, b, Z are in a straight line.

For if Zc is not a tangent to the curve, let Zd touch the curve at d , and draw Pd meeting RQ in m : then RZ is harmonically divided in the points R, m, Q, Z (Ham. P. III. B. V.) and also in the points R, N, Q, Z (as is proved above,) which is impossible; therefore m coincides with N , and d with c , and Zc is a tangent at c .

Again, if the points a, b, Z , are not in a straight line, let Zb be produced to meet the curve in M , and the lines CP, AQ in I and L ; then because of the harmonicals, LZ is harmonically divided in the points L, I, b, Z ; and (Ham. P. III. B. V.) it is also harmonically divided in the points M, I, b, Z , which is impossible, except L and M coincide with the point a , (Ham. B. V. Def. 2, and Cors.); therefore the points a, b, Z , are in a straight line.

Hence it appears, that the lines RQ, ab , and the tangents to the curve at P and c , intersect each other in the same point Z .

Demonstration of the Proposition. Fig. 101, pl. 5.

Let the figure be drawn as before, and let the tangents at the points a, b, c , intersect in E, F , and G . Then, 1st. the proposition affirms that the intersections E, F, G , are in the lines AQ, BR, CP produced. Now, (Ham. P. VIII. B. V.) the intersection A of the tangents RA, PA ; the intersection E of the tangents bE, cE ; and the intersection K , of the lines Rb, cP , are in a straight line; but it has been shewn in the lemma, that the intersection K , is in the line AQ , and therefore the intersection E is in the line AQ produced. In the very same way it may be shewn, that the point F is in the line BR produced, and that the point G is in the line CP produced.

It is manifest, (Ham. *ibid.*) that the lines Rc, Pb , will intersect in AE produced: that the lines Pa, Qc , will intersect in BR produced; and finally, that the lines Ra, Qb , will intersect in GC produced.

2dly. Let the tangents at a, b and c , (fig. 102, pl. 5,) meet the tangents at Q, R , and P , in X, Y , and Z ; then, by the proposition last cited, the intersection X of the tangents aX, QX ; the intersection Y of the tangents bY, RY ; and the intersection of the lines RQ, ab , are in a straight line. But it has been proved, in the cor. to the lemma, that the lines RQ, ab , intersect in the point Z , where the tangents PZ, cZ , intersect, and therefore, the points X, Y, Z , are in a straight line.

The last part of the question is only a particular case of the general proposition enunciated at page 144 of the last number of the Repository, when applied to the conic sections.

Dr. O'Riordan also answered this question.

VII. QUESTION 187, *by* LAPUTIENSIS.

Let two spherical balls given in magnitude and weight, and having their surfaces perfectly polished, be put into a spherical vessel of a given diameter, whose internal surface is also perfectly polished: required the distances of the points of contact from the lowest point of the vessel, when the balls rest in equilibrio?

SOLUTION, *by the Rev. I. TOPLIS.*

Let *OEF* (fig. 103, pl. 5,) be a section of the vessel and spherical balls, perpendicular to the horizon, and passing through their centres; *O* the centre of the vessel, *OF* a vertical line, *OAE*, and *Oae*, lines passing through the centres of the balls to the vessel, *AMa* a line joining their centres. Let *AD* perpendicular to the horizon, represent the weight of the ball *AEM*, and *DE* be drawn parallel, and *AC* meeting it, perpendicular to *OAE*; then the force *AD* may be supposed to be decomposed into the forces *AC*, which accelerates the sphere down the side of the vessel, and *CD*, which acts against the surface of the vessel at *E*. As *Aa* joins the centres of the spheres, it passes through their point of contact *M*, and the force which opposes the descent of the sphere *AEM*, acts in the direction *MA*: let *BA* represent this force which may be decomposed into two others, one parallel to *OAE*, and the other perpendicular to it; the force parallel to *OE*, is opposed by the surface *EF*, and that perpendicular to it, ought to be equal to *AC*, in the case of an equilibrium; therefore these forces may be represented by *BC* and *CA*. If in the sphere *Mea*, *ad* is taken perpendicular to the horizon, and in the same ratio to *AD*, as the weight of the ball *Mea* is to that of *MEA*, and *deb* is drawn parallel, and *ac* perpendicular to *Oe*, the force represented by *ad*, may be decomposed into *cd*, which is opposed by the re-action of the vessel, and *ac* which accelerates the sphere down the vessel. As the force which opposes the rolling of the ball, acts in the direction *Ma*, let it be represented by *ba*, which, in the case of an equilibrium, is equal to *BA*; then *ba*, when decomposed into two forces, one parallel, and the other perpendicular to *Oae*, that perpendicular to it, must evidently be represented by *ca*. As the lines *OE*, *AE*, and *ae* are given, their sums and differences are given, therefore *OA*, *Oa*, and *Aa*, and consequently the triangle *OAA* are given.

Put *Aa* = *a*, *AD* = *W*, *ad* = *w*, and *x* = *AG*; then *aG* = *a* - *x*, and, by similar triangles, *OG* : *AG* :: *AD* : *AB* = *xW* ÷ *OG*; also, *OG* : *aG* :: *ad* : *ab* = (*a* - *x*) *w* ÷ *OG*, and since *AB* = *ab*, we have $xW \div OG = (a-x) w \div OG$,
U 2 or

or $xW = (a - x)w$; therefore $x = a \div (W + w)$. And as in the triangle OAG, AO, AG and $\angle OAG$ are known the remaining sides and angles may be easily found, and in a similar manner the sides and angles of the triangle OGa are known: but the angles EOF and FOe give the distances of the points of contact from the lowest point F. Join FM and in the triangle FGM we know $MG = AM - AG$, $GF = OF - OG$ and $\angle MGF = \angle AGO$, from which we may find MF, or the distance of M from the lowest point.

SECOND SOLUTION, by Dr. O'RIORDAN.

Let OEFe, (fig. 103, pl. 5,) represent a vertical section of the balls and vessel; A and a the centres of the balls; E and e the points of contact with the internal superficies of the vessel: draw Aa, which will manifestly be equal to the sum of the semi-diameters of the two balls, and therefore given; also produce the radii EA, ea, till they meet, which will be in O, the centre of the vessel. Now the common centre of gravity of the two balls, will be a given point in Aa, which suppose G; then through O and G, draw the right line OGF = the radius EO, and join EF, eF.

When the balls are in equilibrio, the line OGF must be perpendicular to the horizon, and F will be the lowest point of the vessel, and EF, eF, the required distances of the points of contact of the balls from the lowest point.

The calculation will be very easy: AOa is a given triangle, and G a given point in the base Aa; whence we find $\angle AOG = \angle FOE$; there is also given $EO = FO$, whence we may easily find EF. And in the same manner eF may be found.

Solutions were also received from Messrs. Amicus and Cunliffe.

VIII. QUESTION 188, by Mr. IVORY.

Two points being given in the same vertical plane, but not in one horizontal line; it is required to determine the position of two inclined planes such, that the time of descent from the higher point to the lower, on these two planes, shall be less than on any other two inclined planes whatsoever?

SOLUTION, by Mr. IVORY, the Proposer.

Let A and B, (fig. 104, pl. 5,) be the two points, and AG, GB, the two inclined planes required: draw the horizontal lines AF and GE, and the vertical lines GD and BF: draw also AH perpendicular to BG.

Let $g = 32\frac{1}{2}$ feet, the accelerating power of gravity: Then the time of descent on the inclined plane, AG will be $= (AG \div GD) \times \sqrt{(BD \div \frac{1}{2}g)} = \tau$:

The

The velocity acquired by the fall $= \sqrt{(2g \times DG)}$:

The velocity, in the direction of the plane GB, $= (GH \div GA) \times \sqrt{(2g \times DG)} = V$.

Let τ' be the time of descent on the inclined plane GB, then $V \times \tau' + (BE \div BG) \times \frac{1}{2}g\tau'^2 = BG$: hence

$$\frac{2}{g} \frac{GH}{GA} \cdot \frac{GB}{BE} \cdot \sqrt{DG} \times (\tau' \sqrt{\frac{1}{2}g}) + (\tau'^2 \sqrt{\frac{1}{2}g})^2 = \frac{BG^2}{BE} \cdot BE: \text{ and}$$

$$\tau' \sqrt{\frac{1}{2}g} = \frac{BG}{BE} \left\{ \sqrt{(BE + \frac{GH^2}{GA^2} \cdot DG)} - \frac{GH}{GA} \sqrt{DG} \right\}. \text{ Therefore}$$

$$(\tau + \tau') \sqrt{\frac{1}{2}g} = \frac{BG}{BE} \left\{ \sqrt{(BE + \frac{GH^2}{GA^2} \cdot DG)} - \frac{GH}{GA} \sqrt{DG} \right\} + \frac{AG}{GD} \sqrt{GD}.$$

If we put $AF = b$, $BF = a$, $GD = x$, and $AD = y$, the expression for $(\tau + \tau') \sqrt{\frac{1}{2}g}$, which is to be a minimum, will become a function of x and y , for which we may write $f(x, y)$: then the equations for determining x and y , in the case of a minimum, will be had by putting the partial fluxions of $f(x, y)$ (the variables being x and y) separately $= 0$: that is the equations

$$\text{will be } \left(\frac{df(x, y)}{dx} \right) = 0, \text{ and } \left(\frac{df(x, y)}{dy} \right) = 0.$$

But as the preceding expression for the wholetime of the fall seems to be somewhat complicated, in order to simplify it, we shall suppose that the falling body passes from the upper plane to the lower, without any loss of velocity, as in the case of a continued curvature; for this purpose, we have only to write 1 in place of $GH \div GA$ in the preceding expression, and we obtain $f(x, y)$

$$= (BG \div BE) (\sqrt{BF} - \sqrt{GD}) + (BA \div GD) \sqrt{GD}:$$

$$\text{And if we put } \epsilon = BA = \sqrt{(x^2 + y^2)}, \epsilon' = BG = \sqrt{\frac{1}{2}(b-y)^2} + (a-x)^2, \text{ then } f(x, y) = \frac{\epsilon'}{a-x} (\sqrt{a-x} - \sqrt{x}) + \frac{\epsilon}{x} \sqrt{x}:$$

$$\text{And remarking that } \left(\frac{d(\frac{\epsilon'}{a-x})}{dy} \right) = - \frac{b-y}{\epsilon'} \cdot \frac{1}{a-x}; \left(\frac{d(\frac{\epsilon'}{a-x})}{dx} \right)$$

$$= \frac{(b-y)^2}{(a-x)^2} \cdot \frac{1}{\epsilon'}; \left(\frac{d(\frac{\epsilon}{x})}{dy} \right) = \frac{y}{\epsilon} \cdot \frac{1}{x}; \left(\frac{d(\frac{\epsilon}{x})}{dx} \right) = - \frac{y^2}{x^2} \times \frac{1}{\epsilon};$$

we shall have according to the method noticed above

$$\left(\frac{df(x, y)}{dy} \right) = - \frac{b-y}{\epsilon} \cdot \frac{\sqrt{a-x} - \sqrt{x}}{a-x} + \frac{y}{\epsilon} \cdot \frac{\sqrt{x}}{x} = 0.$$

$$\left(\frac{df(x, y)}{dx} \right) = \frac{(b-y)^2}{(a-x)^2} \cdot \frac{\sqrt{a-x} - \sqrt{x}}{\epsilon'} - \frac{y^2}{x^2} \cdot \frac{\sqrt{x}}{\epsilon} - \frac{1}{2} \left(\frac{\epsilon'}{a-x} - \frac{\epsilon}{x} \right) \cdot \frac{1}{\sqrt{x}} = 0$$

whence

whence $\frac{b-y}{\epsilon} \cdot \frac{\sqrt{x}}{\sqrt{a+\sqrt{x}}} = \frac{y}{\epsilon}$

$$\frac{b-y}{a-x} \times \frac{b-y}{\epsilon} \times \frac{\sqrt{x}}{\sqrt{a+\sqrt{x}}} - \frac{y}{x} \times \frac{y}{\epsilon} - \frac{1}{2} \left(\frac{\epsilon}{a-x} - \frac{\epsilon}{x} \right) = 0.$$

Let ϕ and ψ denote the angles which the planes AG and GB, make with the horizon; then $\frac{b-y}{a-x} = \frac{\cos \psi}{\sin \psi}$, $\frac{y}{x} = \frac{\cos \phi}{\sin \phi}$,

$$\frac{\epsilon}{a-x} = \frac{1}{\sin \psi}, \frac{\epsilon}{x} = \frac{1}{\sin \phi}, \frac{b-y}{\epsilon} = \cos \psi, \frac{y}{\epsilon} = \cos \phi,$$

we have $\cos \psi \times \frac{\sqrt{x}}{\sqrt{a+\sqrt{x}}} = \cos \phi$,

$$\frac{\cos^2 \psi}{\sin \psi} \times \frac{\sqrt{x}}{\sqrt{a+\sqrt{x}}} - \frac{\cos^2 \phi}{\sin \phi} = \frac{1}{2} \left(\frac{1}{\sin \psi} - \frac{1}{\sin \phi} \right).$$

And if we exterminate $\sqrt{x} \div (\sqrt{a+\sqrt{x}})$, we shall have $2 \cos \phi \times \sin (\phi - \psi) = \sin \phi - \sin \psi$: this equation seems to require $\phi = \psi$, which being inconsistent with the equation, $\cos \psi \times \sqrt{x} \div (\sqrt{a+\sqrt{x}}) = \cos \phi$, will give no solution: but remarking that $\sin \psi = \sin (\phi - (\phi - \psi)) = \sin \phi \cos (\phi - \psi) - \cos \phi \sin (\phi - \psi)$, we get by substitution, $\cos \phi \sin (\phi - \psi)$

$$= \sin \phi (1 - \cos (\phi - \psi)); \text{ whence } \frac{\sin \phi}{\cos \phi} = \tan \phi =$$

$$\frac{\sin (\phi - \psi)}{1 - \cos (\phi - \psi)} = \tan (90^\circ - \frac{\phi - \psi}{2}): \text{ therefore } 3\phi$$

$-\psi = 180^\circ$, and $\cos \psi = -\cos 3\phi$, $\sin \psi = -\sin 3\phi$: or $\cos \psi = 3 \cos \phi - 4 \cos^3 \phi$, and $\sin \psi = 4 \sin^3 \phi - 3 \sin \phi$:

Hence, $\frac{\cos \psi}{\cos \phi} = 3 - 4 \cos^2 \phi = 1 - 2 \cos 2\phi = \frac{\sqrt{a+\sqrt{x}}}{\sqrt{x}}$

put $z = \cos 2\phi$, then $-2z = \sqrt{(a \div x)}$, and $x = a \div 4z^2$.

Again, $AD = \frac{x \cos \phi}{\sin \phi}$, and $GE = \frac{(a-x) \cos \psi}{\sin \psi} = \frac{\cos \phi}{\sin \phi} \times$

$$\frac{(a-x)(3-4\cos^2 \phi)}{4 \sin^2 \phi - 3}: \text{ therefore}$$

$$\frac{\cos \phi}{\sin \phi} \times \left(x + \frac{(a-x)(3-4\cos^2 \phi)}{4 \sin^2 \phi - 3} \right) = b: \text{ And because}$$

$$x = \frac{a}{4z^2}; \frac{\sin \phi}{\cos \phi} = \sqrt{\frac{1-z}{1+z}}; 2 \cos^2 \phi = 1+z, 2 \sin^2 \phi$$

=

$= 1 - z$, we get by substituting and reducing

$$(1 - 2z + 2z^2)^2 (1 + z) = \frac{b^2}{a^3} (4z^4 - 4z^5)$$

whence the problem may be resolved by finding the value of z , or $\cos 2\phi$, in an equation of the 5th degree.

IX. QUESTION 189, by Mr. CUNLIFFE.

Required the sum of the infinite series

$$\frac{1}{2} - \frac{1}{2} \times (\frac{1}{2}) + \frac{1}{4} \times (\frac{1}{2} + \frac{1}{2}) - \frac{1}{8} \times (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) + \frac{1}{16} \times (\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) - \&c \&c$$

SOLUTION, by Dr. O'RIORDAN.

$$\text{Put } z = \frac{x^2}{2} - (\frac{1}{2}) \cdot \frac{x^3}{3} + (\frac{1}{2} + \frac{1}{2}) \cdot \frac{x^4}{4} - (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \cdot \frac{x^5}{5} + \&c.$$

Take the fluxions and divide by $\dot{x}$, then

$$\frac{\dot{z}}{z} = x - \frac{x^2}{2} + (\frac{1}{2} + \frac{1}{2}) \cdot x^2 - (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \cdot x^3 + (\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \cdot x^4 - \&c.$$

Multiply by $1 + x$, and take x^2 from both sides of the equation, then $\frac{\dot{z}}{z} (1 + x) - x^2 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$

Let ϕ be the hyp. log. of $1 + x$, then $\phi = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$ and $\dot{\phi} = \frac{\dot{x}}{1+x}$, and by substitution, $\frac{\dot{z}}{z} = x^2$

$$= \phi, \text{ or } \dot{z} = \phi \dot{\phi} + x^2 \dot{\phi}$$

And taking the fluents, $z = \frac{1}{2}\phi^2 + \phi - x + \frac{1}{2}x^2$, which needs no correction. Let x be taken $= 1$, then ϕ is the log. of 2, and $z = \frac{1}{2}\phi^2 + \phi - \frac{1}{2}$ the sum required.

Mr. Cunliffe, the proposer, by assuming $\frac{\dot{z}}{1+x} \times (x - ax^2 + bx^3 - cx^4 + \&c.) = \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}Qx^4 - \frac{1}{5}Rx^5 + \&c.$ and proceeding as in his solution to question 160, (pa. 87 of the present vol.) obtains the same result as above.

The sum of the series may also be immediately found from Mr. Ivory's solution on page 117 of the present vol. ; for if a be taken $= 1$, then $\frac{1}{2}$ the square of the log. of 2 is $= \frac{1}{2} - (1 + \frac{1}{2}) \cdot \frac{1}{2} + (1 + \frac{1}{2} + \frac{1}{2}) \cdot \frac{1}{4} - (1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \cdot \frac{1}{8} + \&c. =$ to the series in the question, together with the series $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \&c.$ But this last series is evidently $=$ to $\frac{1}{2} - \log. 2$, and

and therefore the sum of the given series is $= \frac{1}{2} \log. 2 - (\frac{1}{2} - \log. 2)$ as before.

The next four Questions are selected from the Cambridge University Calendar for 1806.

X. QUESTION 190, from the CAMB. UN. CAL.

A given cylindric vessel is supplied with water by a cock at a given rate. Then suppose, when the vessel is full, that an aperture of a given area is made in its bottom; what is the lowest point to which the surface of the water in the vessel will descend, and also the time of its descent; the influx of the water by the cock being supposed slower than the efflux when the vessel is full?

SOLUTION, by Mr. CUNLIFFE.

Let h denote the height, and d the diameter of the vessel; A the area of the aperture or hole made in the bottom. Also let x denote any variable height of the water in the vessel whilst running, $g = 16\frac{1}{2}$; and suppose the velocity of the effluent water to be the same as would be generated by gravity in a heavy body falling freely from the height of the surface of the water above the aperture in the bottom, and put $p = .7854$. Then $2\sqrt{gx}$ will express the velocity of the effluent water per second, when the surface is at the height x above the hole in the bottom: and, therefore, $2Agx$ will express the quantity of water issuing per second, when at the height x . Let $2An$ denote the quantity of water per second, supplied by the cock, and then $2A\sqrt{gx} - 2An = 2A \times (\sqrt{gx} - n)$ will express the rate of evacuation of the vessel per second, when the water is at the height x above the aperture;

and therefore $\frac{-d^2 x \dot{x}}{2A(\sqrt{gx} - n)}$ will express the fluxion of the time of evacuation. The fluent of this expression, viz.

$$\frac{nd^2 p}{Ag} \times \text{h. l.} (\sqrt{gx} - n) - \frac{d^2 p \sqrt{gx}}{Ag},$$

expresses the time of evacuating to the height x above the hole. But this expression ought to vanish when $x = h$, therefore the correct expression for the time of evacuating to the height x , will be

$$\frac{d^2 p}{Ag} \times (\sqrt{gh} - \sqrt{gx}) - \frac{nd^2 p}{Ag} \times \text{h. l.} \frac{\sqrt{gh} - n}{\sqrt{gx} - n}.$$

Again

Again it is evident, that the surface of the water in the vessel cannot sink below the point where the rates of the influx and efflux per second, are equal to each other; that is, where $2A\sqrt{gx-n} = 0$, or where $x = n^2 \div g$, in which circumstance the preceding correct expression for the time becomes

$$\frac{d^2 p}{Ag} \times (\sqrt{gh-n}) \div \frac{nd^2 p}{Ag} \times h. \text{ i. } \frac{\sqrt{gh-n}}{0},$$

which is infinite.

Whence it appears, that the surface of the water in the vessel can never arrive at the point, where the rates of influx and efflux are equal to each other.

Dr. O'Riordan answered this question.

XI. QUESTION 191, from the CAMB. UN. CAL.

To this question we have received no answer.

XII. QUESTION 192, from the CAMB. UN. CAL.

A perfectly flexible chain is wound round a cylinder supported with its axis parallel to the horizon. Then, if the weight and dimensions of the cylinder be given, and also the length and weight of the chain; it is required to determine the time in which the chain, impelled by the force of gravity, will unwind itself; a given length being unwound at the commencement of the motion?

SOLUTION, by Mr. CUNLIFFE.

The solution of the following problem will greatly facilitate the answer to the question.

Suppose a cylinder of uniform matter of given dimensions and weight, to be at liberty to turn freely about its axis: it is required to determine what weight must be placed at a point upon the surface of an hollow cylinder of equal dimensions, so as equally to resist the communication of motion by any equal forces acting at the surfaces of the two cylinders, in directions at right angles to the radii.

Let SRC (fig. 105, pl. 5,) represent a section of the cylinder at right angles to its axis, SC a radius thereof; SaA a circle concentric to SRC. Put $SC = r$, $SA = z$, and $c = 3.1416$; and let $2B$ denote the weight of the cylinder. Then $2cz$ = the circumference of the circle SaA, and $2cz\dot{z}$ = the fluxion of the space SaA. And we shall manifestly have $r^2c : 2cz\dot{z} :: 2B :$

$4z\dot{z}B \div r^2 =$ the fluxion of the weight of the circle or cylinder SaA , and the last expression being written for A in the expression, art. 395, page 398, vol. ii, last line but two, Dalby's Course; the expression there becomes $4z^2\dot{z}B \div r^4$, which is the fluxion of the weight to be placed at C : taking the fluents, we have $z^3B \div r^4$, which when $z = r$ becomes B . Hence it appears, that half the given weight of the cylinder being placed at a point in the surface, would equally resist the communication of motion by any force acting at a point C , in a direction CP , at right angles to SC , as the cylinder itself would.

Solution of the question.

Let $2B$ denote the weight of the cylinder, l the length of the chain, and w its weight; and let a denote the length of the part of the chain depending from the cylinder at the commencement of the motion, and let $CP = x$, denote a variable part of the chain depending from the cylinder whilst in motion, and let v denote the corresponding velocity of the end P .

Then we shall obviously have $l : x :: w : wx \div l =$ the weight of the part x , and therefore $2gwx \div l$ expresses the motive force urging the system, when the length x of the chain hangs from the cylinder. But it appears from the preceding problem, that $B + w$ expresses the whole inertia of the system; therefore $2gwx \div l (B + w)$ will express the accelerative force, when the length x of the chain depends from the cylinder.

Then, by a well known theorem, $\frac{2gwx\dot{x}}{l(B+w)} = v\dot{v}$; taking the correct fluents, by making the resulting expression to vanish when $x = a$, we shall have $v^2 = \frac{2gw(x^2 - a^2)}{l(B+w)}$.

Again, $\dot{t} = \frac{\dot{x}}{v} = \sqrt{\frac{l(B+w)}{2gw}} \cdot \frac{\dot{x}}{\sqrt{x^2 - a^2}} =$ the fluxion of the time of unwinding to the length x . And the correct fluents being taken, gives $t = \sqrt{\frac{l(B+w)}{2gw}} \times \text{h.l.} \frac{x + \sqrt{x^2 - a^2}}{a}$, where writing l for x , it becomes $\sqrt{\frac{l(B+w)}{2gw}} \times \text{h.l.} \frac{l + \sqrt{l^2 - a^2}}{a}$ which expresses the time of unwinding the chain.

A solution was also received from Dr. O'Riordan.

XIII.

XIII. QUESTION 193, *from the CAMB. UN. CAL.*

A cylindrical vessel, whose height is equal to its diameter, is filled with water; with what velocity must it be whirled round its axis, that half the water may be thrown out?

SOLUTION, *by Mr. CUNLIFFE.*

I shall first solve the following problem; by means of which, an answer to the question will be readily obtained.

To find the nature of the curve, which the surface of a fluid contained in a cylindrical vessel acquires, whilst the vessel revolves about its axis with an uniform velocity.

Let ABCD, (fig. 106, pl. 5,) represent a section of the cylindrical vessel, by a plane passing through the axis EF; AVB a section of the curved surface which the fluid acquires by the circular motion about EF. Draw the perpendicular ordinate HG, and through the vertex V, draw VR parallel thereto, meeting HR parallel to EF in R.

Put $EB = r$, $c =$ the angular velocity per second, at the distance r , from the axis EF; $VR = GH = y$, $VG = RH = x$, and $g = 16\frac{1}{2}$; then $r : y :: c : cy \div r =$ the angular velocity at R: and hence $c^2y \div r^2 =$ the centrifugal force urging a particle of the fluid at R, and $c^2yy \div r^2 =$ the fluxion of the aggregate of the forces of all the particles in the line VR; the fluent whereof is $c^2y^2 \div 2r^2$, which, from the principles of hydrostatics, must be equal to the pressure of all the particles in the line HR; that is, $c^2y^2 \div 2r^2 = 2gx$, or $4r^2gx \div c^2 = y^2$, an equation expressing the nature of the curve AVHB; and which, from thence appears to be the common parabola.

Solution of the question.

It has just been proved, that the hollow part of the cylinder of water, caused by the whirling motion round the axis, is in the form of a conic paraboloid; and the content of a conic paraboloid is known to be one half of its circumscribing cylinder.

Moreover the equation of the curve was found to be $4r^2gx \div c^2 = y^2$, the quantities x , y , &c. being as specified, and when x becomes $= 2r$, and $y = r$, then $c^2 = 8rg$, or $c = 2\sqrt{(2rg)}$. That is, the velocity per second, with which a point of the cylindrical surface must be whirled about the axis, in the specified case, must be equal to that which an heavy body would acquire, by descending freely from rest by the force of gravity, the whole height of the cylinder,

Q. E. I.

Dr. O'Riordan answered it.

X 2

XIV

XIV. QUESTION 194, by AMICUS.

Of all the ellipses that can be inscribed in the same given quadrilateral, required that which approaches the nearest to a circle?

SOLUTION, by Mr. LOWRY.

Let ACES, (fig. 107, pl. 5,) be the given quadrilateral, *Babd* the inscribed ellipse, *O* its centre, *BOH* a diameter drawn from the point of contact *B*, and *OL* a semi-diameter parallel to *BA*, and which will be conjugate to *BH*. Let *p* be the semi-transverse, and *q* the semi-conjugate axis, then when the ellipse approaches the nearest to a circle, $p \div q$ must be a minimum.

Now $p^2 + q^2 = BO^2 + OL^2 = m^2$, (Emerson on the Ellipse, p. 34.)

and $pq = BO \cdot OL \cdot \sin LOB = n^2$ (Ibid. p. 37.)

or putting $p \div q = r$, we have $p^2 = r^2 q^2$;

and $r^2 q^2 + q^2 = m^2$, or $q^2 = m^2 \div (r^2 + 1)$,

and $r q^2 = n^2$, or $q^2 = n^2 \div r$;

therefore, $m^2 \div (r^2 + 1) = n^2 \div r$, or $r m^2 = n^2 r^2 + n^2$,

and $r = \frac{p}{q} = \frac{m^2}{2n^2} + \sqrt{\left(\frac{m^4}{2n^4} - 1\right)}$, which is to be a minimum.

Produce *AS*, *CE*, till they meet in *V*, and draw *IF* to touch the ellipse at *H*, the extremity of the diameter *BH*, meeting *AS* in *I*, *DS* in *G*, and *CV* in *F*; then *IF* is parallel to *AC*.

Put *AC* = *a*, *AD* = *b*, *AV* = *c*, *AS* = *d*, *SV* = *e*, *AB* = *x*, and *IV* = *y*; then *AB.IH* = *OL*² = *BC.HF*, ibid. p. 47; or *AB* : *BC* :: *HF* : *IH*, and by composition, *AC* : *BC* :: *IF* : *IH* = *IF* $\times$ (*a* - *x*) $\div$ *a*. Also, *AB.IH* = *OL*² = *BD.IG*, or *AB* : *BD* :: *IG* : *IH*; and, by composition, *AD* : *BD* :: *IG* : *IH* = *IG* $\times$ (*b* - *x*) $\div$ *b*; therefore, *IF* $\times$ (*a* - *x*) $\div$ *a* = *IG* $\times$ (*b* - *x*) $\div$ *b*. But by similar triangles,

AV : *AC* :: *IV* : *IF* = *ay* $\div$ *c*,

and *AS* : *AD* :: *IS* : *IG* = *b* (*y* - *e*) $\div$ *d*;

therefore $(a - x) \frac{y}{c} = (b - x) \left\{ \frac{y - e}{d} \right\}$.

Draw *IW* parallel to *BH*, then *IH* being = (*a* - *x*) *y* $\div$ *c*, *AW* is = *AB* - *IH* = *x* - (*a* - *x*) *y* $\div$ *c*.

Put *s* = the sine, and *t* = the cosine of the given angle *IAW*, then by trigonometry, *IW*² (= *BH*²) = *AI*² + *AW*² - 2*t*.

AI.AW; but *AI*² = (*c* - *y*)², *AW*² = $\left\{ x - (a - x)y \div c \right\}^2$,

2*t*.*AI.IW* = 2*t* (*c* - *y*) $\left\{ x - (a - x)y \div c \right\}$, 4*OL*² = 4*AB*

$\times$ *IH* = 4*y* (*a* - *x*) *y* $\div$ *c*; and collecting the terms, we have

*BH*² + 4*OL*² = 4*m*² = (*c* - *y*)² + $\left\{ x + (a - x)y \div c \right\}^2$ - 2*t* (*c* - *y*) $\left\{ x - (a - x)y \div c \right\}$.

Agg'u.

Again, the sine of the angle AWI, or of BOL, is $= s(c-y) \div IW = \frac{1}{2}s(c-y) \div OB$; therefore $BO.OL.\sin \angle BOL = \frac{1}{2}s(c-y) \vee \{(a-x)y \div c\} = n^2$; these values of m^2 and n^2 being substituted in the equation $\frac{p}{q} = \frac{m^2}{2n^2} + \vee(\frac{m^4}{2n^4} - 1)$,

we have an expression for $p \div q$ in terms of x , y , and known quantities. By making the fluxion of this expression $=$ to 0, we obtain a value of $\dot{x} \div \dot{y}$; and by taking the fluxion of the equation $(a-x)y \div c = (b-x)(y-c) \div d$, we get another value of $\dot{x} \div \dot{y}$, which being equated with the former value, we obtain an equation expressing the relation of x and y , and eliminating x or y , by means of the equation $(a-x)y \div c = (b-x)(y-c) \div d$, we finally get the values of x and y in known quantities, and then the ellipse may easily be described by known methods.

XV. QUESTION 195, by Mr. I. R.

A parabola, and a point within it, being given in position, it is required to describe another parabola, that shall pass through the given point, touch the given parabola in a given point, and have its axis parallel to the axis of the given parabola.

SOLUTION, by Mr. LOWRY.

Let ABC, (fig. 108, pl. 5,) be the given parabola, P the given point within it, and PDQ the required parabola, passing through P, touching the given parabola in the given point Q, and having its axis parallel to the axis of the given parabola. Bisect PQ in I, and draw IE parallel to the axes of the parabolas meeting the curve in D, and the common tangent to the parabolas at Q in E; then, since I is a given point, and the tangent QE is given by position, the line IE is given in magnitude; and, therefore, ID ($=$ DE, Em. Par. 20,) is also given in magnitude: Hence the ordinate PQ, and the diameter ID, are given to describe the parabola.

Or we may determine the focus and directrix of the parabola, by the following method: let f be the focus, join Pf, Qf, and PE, and draw the diameters PB, QF; then EP is a tangent to the parabola at P, (Em. Par. 20, Cor. 2,) and bisects the angle BPf (96); also the tangent QE bisects the angle FQf; therefore if Pf and Qf be drawn to make the angles EPf, EQf equal to the given angles BPE, EQF respectively, their intersection will determine the point f ; and then, if PL be taken equal to Pf, a line drawn through L perpendicular to PL, will be the directrix.

When

When P is the focus of the given parabola, (fig. 109, pl. 5,) and B the vertex, EP bisects the angle BPQ , and EQ bisects the angle PQF , (Em. Par. 8, Cor. 3;) therefore the point f is in the line PQ , and may be determined by drawing Ef perpendicular to PQ .

Let the tangent at the vertex of the diameter ID , meet PE in H ; draw HK parallel to PB , and join HB , BE ; then $PH = HE$ (20,) and because HK is parallel to PB , the angle PKH is $= BPH = HPK$; therefore the triangle PHK is isosceles, and $PK = HK = ID$. Again, because PE bisects the angle BPQ , BE is a tangent to the parabola at B , (36), and therefore perpendicular to PB ; and since EP is bisected in H , the point H is the centre of a circle passing through the points P , B , E ; therefore the angle BHE is $=$ to twice the angle BPH or $=$ to the angle BPQ . From hence we may infer, that if FQ be drawn from the focus, to any point Q , in the parabola ABC , and PE bisect the angle BPQ , and BH be drawn to make the angle $BHE = BPQ$, and HK be parallel to PB ; then PQ is $=$ to four times PK , or to four times ID ; and the parabola described with the diameter ID , and ordinate PQ , will touch the parabola ABC at Q .

This conclusion immediately leads to the solution of the following problem: To determine the curve that touches all the parabolas described by a projectile from the same point, with the same initial velocity at different angles of elevation in the same vertical plane.

Let P be the initial position of the projectile, PE its direction, and PB the impetus, or the height to which the projectile would ascend in a vertical direction, with the given initial velocity.

Draw BH to make the angle $BHE = BPQ$, and draw HK parallel to PB ; then it is shewn, by the writers on projectiles, (see Dalby's Course of Mathematics, Vol. 2, Art. 335,) that the parabola described by the projectile, will meet the line PQ , at a distance from P equal to four times PK : but it is proved above, that four times PK is equal to PQ , therefore the parabola described by the projectile, meets PQ in the point Q , where that line meets the parabola ABC . Also the greatest height of the projectile, or the diameter ID , is equal to HK , (ibid;) therefore the parabola described by the projectile, is the same as the parabola PDQ described above, and consequently the projectile will touch the parabola ABC at Q . And as a like point of contact may be found for any other angle of elevation, it is obvious that each parabola described by the projectile, will touch the parabola ABC ; and, consequently, the curve required is the parabola ABC , whose *latus rectum* is $=$ to four times the impetus.

COR. I. The triangles BPE , PEf , are evidently equal in every

every respect; therefore, $Pf = PB$, and BE is the directrix of all the parabolas described by the projectile.

COR. 2. A circle described with the centre P , and radius PB , or (Pf) will be the locus of the foci of all these parabolas.

COR. 3. If we conceive the parabola ABC , to revolve round the axis PB , so as to generate a parabolic superficies, it is obvious that this superficies will touch all the parabolas that can possibly be described by the projectile from P , with the given impetus, and in any direction whatever: and the locus of the foci will then be a spherical superficies.

Solutions were also received from Messrs. Cunliffe & O'Riordan.

XVI. QUESTION 196, by Mr. WILLIAM SLENDER.

Let there be a right line and a point given by position in the same vertical plane: and suppose a flexible string of a given length, considered as without weight, to be fastened at one end to the given point, and a small heavy body or bead to slide freely thereon; required the locus of the head, whilst the other end of the string is carried slowly along the right line given by position?

SOLUTION, by Mr. CUNLIFFE.

Let P , (fig. 110, pl. 5,) be the given point to which one end of the string PQR is fastened; SR the right line given by position, in the same vertical plane with P , and along which the other end R of the string, is slowly carried: also, let Q be the place of the bead corresponding with the point R .

Produce RS to meet PA , parallel to the horizon, in A , and draw PS perpendicular to PA , cutting RS in S . Produce PQ to r , so that $Qr = QR$, and draw the lines nr , NQ , and mR , parallel to PA , and terminating in PS in the points n , N , and m respectively: also, join Rr , and produce NQ to meet it in v .

Now the end R of the string is, by the question supposed to move very slowly along SR , and consequently, the bead Q will slide slowly along the string; therefore the bead Q may be imagined to be in equilibrio at every corresponding point R , upon the line SR . In the last mentioned circumstance it is well known, that $\angle PQN = \angle rQv = \angle RQv$, therefore Qv always bisects, and is perpendicular to Rr , and consequently Rr is parallel to PSN .

Put $PS = a$, $PA = b$, the length of the string $PQR = Pr = l$, $PQ = r$; tang. $\angle NPQ = t$, to radius 1, then its secant = $\sqrt{1+t^2}$

$\sqrt{(1+t^2)}$, and by the principles of trigonometry,
 $\sqrt{(1+t^2)} : t :: l : lt \div \sqrt{(1+t^2)} = nr = mR$, $\sqrt{(1+t^2)} : 1 :: l : l \div \sqrt{(1+t^2)} = Pn$; also, $\sqrt{(1+t^2)} : 1 :: r : r \div \sqrt{(1+t^2)} = PN$. And because Nv bisects Rr , or $Nm = Nn$, $Pm = 2PN - Pn$, and therefore, $Sm = 2PN - Pn - PS = (2r - l) \div \sqrt{(1+t^2)} - a = \left\{ 2r - l - a \sqrt{(1+t^2)} \right\} \div \sqrt{(1+t^2)}$. Again, by reason of similar triangles, APS, RmS , $PA : PS :: mR : Sm$, or

$b : a :: lt \div \sqrt{(1+t^2)} : \left\{ 2r - l - a \sqrt{(1+t^2)} \right\} \div \sqrt{(1+t^2)}$; whence
 $r = \frac{1}{2}alt \div b + \frac{1}{2}a \sqrt{(1+t^2)} + \frac{1}{2}l = \frac{1}{2}a (nt + \sqrt{(1+t^2)}) + \frac{1}{2}l$, or
 $r^2 = \frac{1}{4}a^2 (n^2 t^2 + 2nt \sqrt{(1+t^2)} + 1 + t^2) + \frac{1}{2}alt (nt + \sqrt{(1+t^2)}) + \frac{1}{4}l^2$,
 where $n = l \div b$.

When $t = 0$, $r = \frac{1}{2}(l + a)$, which determines the point where the curve cuts PS .

From the preceding equation of the curve, an equation expressed in terms of its own co-ordinates, with respect to the line PSN , as an axis may be easily derived. In order to this end, put $PN = x$, and $NQ = y$, then $r = \sqrt{(x^2 + y^2)}$, and we shall very evidently have

$x : y :: 1 : y \div x = t$, $x : \sqrt{(x^2 + y^2)} :: 1 : \sqrt{(x^2 + y^2)} \div x = \sqrt{(1+t^2)}$;
 Now by means of these exterminating r and t out of the said equation, and it becomes $\sqrt{(x^2 + y^2)} = \frac{a}{2} \left(\frac{ny}{x} + \frac{\sqrt{(x^2 + y^2)}}{x} \right) + \frac{l}{2}$; whence $(2x - a) \sqrt{(x^2 + y^2)} = any + lx$, which is an equation for a line of the fourth order.

The area may be determined as follows:

Let z denote the circular arc, radius 1, and centre P_2 intercepted between PS and PQ ; then we shall have $1 : r :: \dot{z} : r\dot{z}$ = the fluxion of the circular arc, radius r , and centre P , intercepted between PS and PQ , and hence $r \cdot \frac{r\dot{z}}{2} = \frac{r^2\dot{z}}{2} = \frac{a^2}{8} \times \frac{\dot{t}}{1+t^2} (n^2 t^2 + 2nt \sqrt{(1+t^2)} + 1 + t^2) + \frac{al}{4} \cdot \frac{\dot{t}}{1+t^2} \times (nt + \sqrt{(1+t^2)}) + \frac{l^2}{8} \cdot \frac{\dot{t}}{1+t^2}$ = the fluxion of the area of the space comprehended within the lines PS , PQ , and the curve; and the correct fluent of this expression is
 $\frac{l^2 - a^2 n^2}{8} \cdot z + \frac{a^2 t}{8} (n^2 + 1) + \frac{a^2 n}{4} \sqrt{(1+t^2)} + \frac{aln}{8} \times \text{h. l. } (1+t^2)$
 $+ \frac{al}{4} \times \text{h. l. } (t + \sqrt{(1+t^2)}) - \frac{a^2 n}{4}$.

When

When $l = an = al \div b$, the quantity $\frac{l^2 - a^2 n^2}{8} \times z$, will

manifestly disappear out of the general expression for the area, which indicates, that when the line SR makes an angle of 45° with the horizon, the area is expressible independent of circular arcs; that is, the area in that case will be expressed by algebraic quantities and hyperbolic logarithms.

*Solutions to this question were also received from Messrs.
O'Riordan and Swale.*

XVII, QUESTION 197, by Mr. IVORY.

If there be two concentric ellipses, similar to one another, and similarly posited, the proportion of the axes being that of $\sqrt{2}$ to 1; then, if from any point in the periphery of the inner ellipse, any right line be drawn to terminate in the periphery of the outer ellipse, the rectangle under the segments of that line, is equal to the square of that semi-diameter of the inner ellipse, which is parallel to the right line. Required the demonstration?

SOLUTION, by Mr. I. H. SWALE.

When the axes of the ellipses have the proportion in the question, the outer ellipse will be the locus of the angles of the parallelogram, formed by drawing tangents at the extremities of any two conjugate diameters of the inner ellipse, as is shewn at page 237, vol. 1. Hence, if any line CPD, (fig. 111, pl. 5,) be drawn to touch the inner ellipse at P, and meet the outer one in C and D, and the diameter AB, of the inner ellipse, be drawn parallel to CD; then AB will be = CD, and $AQ = QB = CP = DP$.

Now let any line be drawn through P, to meet the outer ellipse in K and L, and let QR be the semi-diameter, parallel to KL: Then, by a well known property of the sections, the rectangle CP.PD is to the rectangle KP.PL, as the square of the semi-diameter of the outer ellipse parallel to CD, to the square of the semi-diameter of the same ellipse parallel to KL; that is, because of the similarity of the ellipses, as AQ^2 to QR^2 : but CP.PD is = to AQ^2 , therefore KP.PL is = to QR^2 .

This property is remarkable for its analogy to a property of the asymptotes of a hyperbola; for as the outer ellipse is the locus of the angles of all the parallelograms circumscribing the inner ellipse; so the asymptotes of a hyperbola are the loci of the angles of all the parallelograms formed by tangents drawn to the opposite and conjugate hyperbolas: And if any line be drawn through a point P in a hyperbola, to meet the asymptotes in K

and L, the rectangle KP.PL is equal to the square of the semi-diameter parallel to CL, as in the ellipse.

It was also answered by Dr. O'Riordan.

XVIII. QUESTION 198, by MECHANICUS.

Required the nature of the curve along which a heavy body descending by the force of gravity, with a given initial celerity, shall prefs upon the curve at any point with a force reciprocally proportional to the radius of curvature at that point ?

SOLUTION, by Mr. LOWRY.

Suppose that ABC, (fig. 112, pl. 5,) is the curve required, and that the body begins to descend from B, with the same velocity that it would have acquired by falling through the vertical height FB.

Let C be any point in the curve, and draw BD perpendicular to the vertical line CD. Put $BD = x$, $DC = y$, arch $BC = s$, $FB = a$, the radius of curvature at C = r , the velocity of the body at C = v , and the accelerative force of gravity = g .

Then the whole pressure on the curve at C, (the mass of the body being unit,) is $g \times \frac{\dot{x}}{s} + \frac{v^2}{r}$, the part $g \times \frac{\dot{x}}{s}$ being

the normal pressure arising from gravity, and the part $\frac{v^2}{r}$, the pressure arising from the centrifugal force; but since the velocity of the body at C, is the same as would be acquired by descending through the height BF + CD or $a + y$, therefore $v^2 = 2g(a + y)$, and the whole pressure = $g \left(\frac{\dot{x}}{s} + \frac{2(a + y)}{r} \right)$.

But by the question, the pressure is always reciprocally proportional to the radius of curvature, or = to $\frac{n}{r}$, where n is a constant quantity; therefore $g \left(\frac{\dot{x}}{s} + \frac{2(a + y)}{r} \right) = \frac{n}{r}$, or putting $2a - m = 2b$, $m = n \div g$; and multiplying by r , we have $r \times \frac{\dot{x}}{s} + 2(b + y) = 0$.

Now when s is constant, the general expression for the radius of

of curvature, is $\frac{\dot{s}\ddot{y}}{x}$, which being put in the last equation instead

of r , we have $\frac{\dot{s}\ddot{y}}{x} + 2(b+y) = 0$, or $x\ddot{y} + 2(b+y)\dot{x} = 0$;

and dividing by $2(b+y)^{\frac{1}{2}}$, we have

$$\frac{\dot{s}\ddot{y}}{2(b+y)^{\frac{1}{2}}} + (b+y)^{\frac{1}{2}}\dot{x} = 0.$$

Take the fluents, supposing $\dot{y}$ to be constant, and then $(b+y)^{\frac{1}{2}}\dot{x} = A^{\frac{1}{2}}\dot{y}$, where $A^{\frac{1}{2}}$ is a constant quantity necessary for completing the integral, and which must be determined from the particular data of the problem.

Let the last equation be divided by $(b+y)^{\frac{1}{2}}$, and the fluent taken, then $x = 2A^{\frac{1}{2}}(b+y) + C$, an equation to the common parabola.

In this equation, when $y = 0$, x is $= 2A^{\frac{1}{2}}b^{\frac{1}{2}} + C$, but x should then be $= 0$, therefore C is $= -2A^{\frac{1}{2}}b^{\frac{1}{2}}$, and $x = 2A^{\frac{1}{2}}\{(b+y)^{\frac{1}{2}} - b^{\frac{1}{2}}\}$, or $(x + 2A^{\frac{1}{2}}b^{\frac{1}{2}})^2 = 4A(b+y)$.

Hence it is manifest, that if $4A$ be the parameter to the axis AE , and BH be taken $=$ to b , the line HA drawn parallel to CE , will meet the parabola at A , the vertex of the axis.

To determine the value of the parameter, let AI be taken on the axis $=$ to HF or $\frac{1}{2}m$; then it is obvious that the velocity, (and consequently the pressure) at any point of the curve, will be the same, whether the body begins to descend from A , with the velocity due to the height AI , or from B , with the velocity due to the height FB .

Now $\frac{\dot{x}}{s}$ is $=$ the sine of the angle which the tangent makes with the axis of the curve, and therefore at the vertex of the parabola $\frac{\dot{x}}{s}$ is $= 1$, and the normal pressure $= g$. Also the radius of curvature at the vertex, is equal to half the parameter or $\frac{1}{2}p$, and the centrifugal force $= \frac{v^2}{r} = \frac{2g \times AI}{\frac{1}{2}p} = \frac{2gm}{p}$; but since

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the centre of curvature is on the contrary side of the curve with the body, it is evident that the centrifugal pressure is opposed to the normal pressure, and consequently the whole pressure at A, is $g - \frac{2gm}{p}$, which by the question is $\frac{n}{r} = \frac{2n}{p}$; therefore p is $= 4m$.

The truth of this conclusion may easily be verified by a synthetical process; for let the abscissa AE, be put $= v$,

and the ordinate CE $= u$; then u^2 is $= pv$, $\dot{u} = \frac{p^{\frac{1}{2}} \dot{v}}{2v^{\frac{1}{2}}}$, $\dot{s} = \frac{\dot{v}}{2} \sqrt{\frac{4v+p}{v}}$, and $\frac{\dot{u}}{\dot{s}} = \frac{p^{\frac{1}{2}}}{\sqrt{(4v+p)}} = \frac{p^{\frac{1}{2}}(4v+p)}{(4v+p)^{\frac{3}{2}}}$.

But (Simpson's Fluxions, pa. 76, vol. 1.) $\frac{(4v+p)^{\frac{3}{2}}}{2p^{\frac{1}{2}}} = r$;

therefore, $\frac{\dot{u}}{\dot{s}} = \frac{4v+p}{2r}$, and the normal pressure is $=$

$\frac{g(4v+p)}{2r}$; Also the centrifugal force $\frac{v^2}{r} = \frac{2g(v+\frac{1}{2}m)}{r}$
 $= \frac{g(4v+2m)}{2r}$; therefore the whole pressure is $=$

$\frac{g(4v+p)}{2r} - \frac{g(4v+2m)}{2r} = \frac{g(p-2m)}{2r} = \frac{gm}{r}$.

It is evident from hence, that if the body begins to descend from A, with any given velocity, the pressure at any point of the parabola, will be equal to a certain quantity divided by the radius of curvature at that point.

Messrs. O'Riordan and Swale answered this question.

XIX. QUESTION 199, by Mr. CUNLIFFE.

Find a series of numbers, every one of which is divisible into three such parts, that if to the square of each part the product of the other two be added, the three sums thence arising shall all be rational squares?

SOLUTION, by Mr. CUNLIFFE, the Proposer.

Let n denote a number capable of being divided according to the conditions of the question; and let x , y , and z , denote the required

required parts thereof. Then by the question, $yz + x^2$, $xz + y^2$, and $xy + z^2$ are all to be rational squares: put $z = 4(x + y)$, and then

$$yz + x^2 = x^2 + 4xy + 4y^2 = (x + 2y)^2 = \text{a square,}$$

$$xz + y^2 = 4x^2 + 4xy + y^2 = (2x + y)^2 = \text{a square,}$$

$$xy + z^2 = 16x^2 + 33xy + 16y^2 = \text{a square.}$$

Assume $4ry - 4x$, for the root of the last square, that is, put $16x^2 + 33xy + 16y^2 = (4ry - 4x)^2 = 16r^2y^2 - 32rxy + 16x^2$, which gives $x = 16y(r^2 - 1) \div (32r + 33)$, and $z = 4(x + y) = 4y(16r^2 + 32r + 17) \div (32r + 33)$; also $x + y + z = 5(x + y) = 5y(16r^2 + 32r + 17) \div (32r + 33)$, which put = n : whence

$$y = n(32r + 33) \div 5(16r^2 + 32r + 17),$$

$$x = 16y(r^2 - 1) \div (32r + 33) = 16n(r^2 - 1) \div 5(16r^2 + 32r + 17),$$

$$\text{and } z = 4y(16r^2 + 32r + 17) \div (32r + 33) = \frac{4}{5}n.$$

Hence it appears, that any number whatever, is capable of being divided according to the conditions of the question, and consequently, any series of numbers may be so divided, which shews the question to be very unlimited indeed. But if we confine our attention to such a series of numbers as are capable of being divided into three integral parts of the kind mentioned, we shall find the scope of the question more circumscribed. Numbers of the last mentioned kind are contained in the form $n = 5(16r^2 + 32r + 17)$, where r is any whole number at pleasure; and the three required parts of this number are $x = 16(r^2 - 1)$, $y = 32r + 33$, and $z = 4(16r^2 + 32r + 17)$, the reason of which is obvious from barely inspecting the foregoing conclusions.

Dr. O'Riordan also answered this question.

XX. QUESTION 200, by HYPATIA.

Required the nature of the curve, that is, the locus of the point in which a perpendicular from the centre of an ellipse meets a tangent to the ellipse. Required also the rectification of the curve and its quadrature?

SOLUTION, by Mr. LOWRY.

Let AB, (fig. 113, pl. 5,) be the transverse, and DH the conjugate axis, C the centre, and F one of the foci of the ellipse. Also let P be the point where a perpendicular from the centre, meets a tangent drawn to the ellipse, at any point Q. Through the focus F, draw FE perpendicular to the tangent, meeting it in E; complete the rectangle CPEG, and draw CE; then CE is equal to the semi-transverse CB. (Emerson's Conic Sections, B. I. P. 20. Cor. 2.)

Put

Put $CB = a$, $CF = e$, $CP = y$, the angle BCP , (or its measure, the arc ON , reckoned on the circle whose radius CN is 1,) $= x$, and the area of the space $BCPB = s$.

Then because CP is parallel to GE , the angle CFG is equal to the angle $BCP = x$; therefore $CG = CF \sin x = e \sin x$, and $EG (CP) = \sqrt{(CE^2 - CG^2)} = \sqrt{(a^2 - e^2 \sin^2 x)}$,

$$\text{or } y = \sqrt{(a^2 - e^2 \sin^2 x)},$$

the equation of the curve expressed in terms of the variable line CP , and the variable angle BCP .

From this equation it appears, that when x is either $=$ to 0, or to two right angles, y is $=$ to a , and when x is either a right angle, or three right angles, $\sin^2 x$ is $= 1$, and $y = \sqrt{(a^2 - e^2)} = CD$ or CH . Therefore it is evident that the curve will touch the ellipse in the points B , D , A , H , and that it will consist of four quadrantal spaces, equal and similar to the space $CDPB$.

Let p be a point in the curve, indefinitely near to P , and draw Cp , meeting the circle, whose radius is CN in n : also draw pI perpendicular to CP .

Then because On is the fluxion of the angle BCP , or $= \dot{x}$, pI is $= PC \times \dot{x}$; therefore the triangle CpP ($= \frac{1}{2}PC \cdot pI$) or the fluxion of the area $BCPB$ is $= \frac{1}{2}PC \times \dot{x} = \frac{1}{2}y\dot{x} = \frac{1}{2}(a^2 - e^2 \sin^2 x) \dot{x}$; that is $\dot{s} = \frac{1}{2}a^2\dot{x} - \frac{1}{2}e^2 \sin^2 x \dot{x}$, and taking the fluents, we have

$$s = \frac{1}{2}a^2 x - \frac{1}{4}e^2 x - \frac{1}{8}e^2 \sin 2x,$$

which needs no correction.

And when x is a right angle, s is $= \frac{1}{2}p(a^2 - \frac{1}{2}e^2)$, p being the fourth part of the periphery of a circle whose radius is 1, therefore the whole area, or four times the space $CDPB$ is

$$= 2p(a^2 - \frac{1}{2}e^2).$$

Again, let the arc PB be denoted by z , and then $\dot{z}$ is $= \sqrt{(pI^2 + IP^2)}$: But pI is $= y\dot{x} = \dot{x}\sqrt{(a^2 - e^2 \sin^2 x)} = \dot{x}(a^2 - e^2 \sin^2 x) \div \sqrt{(a^2 - e^2 \sin^2 x)}$, and $IP = \dot{y} = \dot{x}(e^2 \sin x \cos x) \div \sqrt{(a^2 - e^2 \sin^2 x)}$; and substituting these values of pI and IP in the expression for $\dot{z}$, we have, after proper reduction,

$$\dot{z} = \frac{\dot{x} \sqrt{\{a^4 - (2a^2 - e^2)e^2 \sin^2 x\}}}{\sqrt{(a^2 - e^2 \sin^2 x)}};$$

or putting $q = a^4 \div (2a^2 - e^2)e^2$, $p = a^2 \div e^2$, and $r^2 = 2a^2 - e^2$

$$\dot{z} = \frac{r\dot{x}\sqrt{(q - \sin^2 x)}}{\sqrt{(p - \sin^2 x)}}.$$

The fluent of this expression, may, in some particular cases, be expressed by elliptic arcs, but in general it is included in the formula H , or the third and most complex species of elliptic transcendentials, as will easily appear by making the proper transformation according to the method of Le Gendre.

If w be put $= q - \sin^2 x$, the expression for $\dot{z}$ becomes

$$\frac{rw^{\frac{1}{2}}\dot{z}}{\sqrt{(p-q+w)}}. \text{ Or since } \sin x = \sqrt{(q-w)}, \cos x = \sqrt{(1-q+w)}, \text{ and } \dot{z} = \frac{\text{flux. } \sin x}{\cos x} = \frac{-\dot{w}}{2\sqrt{\{(q-w)(1-q+w)\}}},$$

we have

$$\dot{z} = \frac{-\frac{1}{2}rw^{\frac{1}{2}}\dot{w}}{\sqrt{(p-q+w)}\sqrt{\{(q-w)(1-q+w)\}}}.$$

This last expression agrees with formula 55, table 12, of Landen's Appendix, where the ingenious author has shewn how it may be resolved into other expressions that are more simple; but the integration by this method is exceedingly troublesome. Perhaps the easiest way of computing the fluent is at once to resolve the expression into a series without having recourse to any particular mode of transformation.

Messrs. Cunliffe, O'Riordan, and Swale, also answered it.

XXI, QUESTION 201, by Mr. IVORY.

If straight lines be inflected from the extremities of the transverse axis of an ellipse, or hyperbola, to any point in the curve, and perpendiculars, drawn to the inflected lines from the same point, be produced to cut the same axis, the part of the axis intercepted by the perpendiculars, will be constantly of the same magnitude. Required the demonstration?

SOLUTION, by Mr. I. H. SWALE.

Fig. 114, 115, pl. 5. Let AB be the transverse axis of the ellipse or hyperbola, and let the lines AP, BP be inflected to any point P in the curve. Draw PE perpendicular to AP, and PD perpendicular to BP, then by the question, ED is a constant magnitude.

Draw PC perpendicular to AB, then (Euc. VI. 8.)

$PC^2 = AC.CE = BC.CD$, therefore $AC : BC :: CD : CE$, and by composition or division, according as the curve is an ellipse or hyperbola, $AB : BC :: ED : CE$, or $AB : ED :: BC : CE :: AC.BC : PC^2 (AC.CE)$.

But by a well known property of the sections,

$AB : \text{latus rectum} :: AC.BC : PC^2$;

Therefore DE is equal to the latus rectum.

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A similar property belongs to the parabola; for if a line be drawn from the vertex A, to any point P in the curve, and PE be drawn perpendicular to AP, and PD perpendicular to the diameter at P; then the distance ED, on the axis, is equal to the latus rectum of the parabola.

If the lines are drawn from the extremities of the conjugate axis, to any point of the ellipse, or conjugate hyperbolas; then DE will still be a constant magnitude, being equal to a third proportional to the conjugate and transverse axes; and the demonstration of this case will be precisely the same as the preceding.

This question was also answered by Dr. O'Riordan.

XXII. QUESTION 202, by Mr. WALLACE, R. M. College.

Find the length of a curve, the nature of which is expressed by the equation

$$e^y = \frac{e^x + 1}{e^x - 1},$$

where x and y denote the co-ordinates, and e the number of which the hyperbolic logarithm is unity?

SOLUTION, by Mr. IVORY.

Put $z = e^x$, then $e^y = \frac{e^x + 1}{e^x - 1} = \frac{z + 1}{z - 1}$: therefore x

$= \log. z$, and $y = \log. \frac{z + 1}{z - 1}$; or, if we put $u = \frac{z + 1}{z - 1}$,

then $(z - 1)(u - 1) = 2$, and $x = \log. z$, and $y = \log. u$. When z increases from 1 to ∞ , it is obvious that u decreases from ∞ to 1: therefore when x , or $\log. z$, increases from 0 to ∞ , it follows that y , or $\log. u$, decreases from ∞ to 0. Hence it is manifest, that the curve has two asymptotes, AB and AC, (fig. 116, pl. 5,) passing through the origin of the co-ordinates, and parallel to them. In the case when $z = u$, we have $z = u = \sqrt{2 + 1}$, and $x = y = \log. (\sqrt{2 + 1})$; which determines the point D where the curve cuts the line AD that bisects the angle of the asymptotes. And because z and u are interchangeable in the equation $(z - 1)(u - 1) = 2$, therefore the two branches of the curve, on opposite sides of this line, are similar and equal.

If, in the equation $(z - 1)(u - 1) = 2$, z be taken less than 1, (in which case x , or $\log. z$, would be negative), then u would be negative, and so y , or $\log. u$ would be the logarithm of

of a negative quantity, which is an absurd expression: therefore, although four curves, all similar and equal, may be described in the four angles of the asymptotes, yet these four curves are not connected by the law of continuity as two opposite hyperbolas are. For the equation that applies to one of the four curves, does not apply to any of the rest, by giving to the variable quantities of the equation all possible values from 0 to $+\infty$ on one hand, and from 0 to $-\infty$ on the other hand, while the origin of these variable quantities remains fixt.

The area ABMN, or curvilinear space between the asymptote and the ordinate MN, or y , $= \int y dx = \int \frac{dz}{z} \log. \frac{z+1}{z-1}$

$$= \int \frac{dz}{z} \log. \frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} = C - 2\left(\frac{1}{z} + \frac{1}{3^2 \cdot z^3} + \frac{1}{5^2 \cdot z^5}\right.$$

$+ \&c.$): where $C = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.\right)$.

The whole curvilinear space between the curve and the two asymptotes, is $= C = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.\right)$: there-

fore half this space, or the space ABND, bounded by the line that bisects the angle of the asymptotes, either asymptote and

the corresponding branch of the curve, is $= \frac{1}{2}C = 1 + \frac{1}{3^2}$

$+ \frac{1}{5^2} + \frac{1}{7^2} + \&c.$ to this let the area of the right-angled

triangle ADE, comprehended by the co-ordinates that correspond to the middle point of the curve, or $\frac{1}{2} \log.^2 (\sqrt{2} + 1)$, be added, and the whole area, between the asymptote and the ordinate DE, drawn parallel to the asymptote from the middle point of the curve, is $= \frac{1}{2}C + \frac{1}{2} \log.^2 (\sqrt{2} + 1)$: but when

$z = \sqrt{2} + 1$, then $\frac{1}{z} = \sqrt{2} - 1 = a$, and the area ABDE

$= C - 2\left(a + \frac{a^3}{3^2} + \frac{a^5}{5^2} + \&c.\right)$: therefore, by equating

the two values of the area ABDE, we get, $1 + \frac{1}{3^2} + \frac{1}{5^2}$

$+ \&c. = 2\left(a + \frac{a^3}{3^2} + \frac{a^5}{5^2} + \&c.\right) + \frac{1}{2} \log.^2 (\sqrt{2} + 1)$.

(Vide Landen's Memoirs, pages 116, 117).

The length of an arch of the curve, as ND, $= \int \sqrt{(dx^2 + dy^2)}$
 $= \int \sqrt{\left(\frac{dz^2}{z^2} + \frac{4dz^2}{(z^2 - 1)^2}\right)} = \int -dz \cdot \frac{z^2 + 1}{z^2 - 1}$, (because
the arch DN increases when z decreases) $= -\int dz + \int \frac{dz}{z + 1}$
 $= -\int \frac{dz}{z - 1} = C - z + \log. \frac{z + 1}{z - 1} = \sqrt{2 + 1} - \log.$
 $(\sqrt{2 + 1}) - e^x + y$. Hence the difference of the ordinate
MN, (or y), and the arch ND $= \sqrt{2 + 1} - \log. (\sqrt{2 + 1})$
 $- z = \sqrt{2 + 1} - \log. (\sqrt{2 + 1}) - e^x$: and the excess of
the whole curve Dn above the asymptote AB, both continued
indefinitely, $= \sqrt{2 + 1} - \log. (\sqrt{2 + 1}) - 1 = \sqrt{2} - \log.$
 $(\sqrt{2 + 1})$.

If we take $z = \sqrt{2 + 1} - \log. (\sqrt{2 + 1})$, or $x = \log.$
 $\left\{ \sqrt{2 + 1} - \log. (\sqrt{2 + 1}) \right\}$, a point will be determined in
the curve such, that the arch DN = ordinate NM.

Messrs. O'Riordan and Toplis sent solutions to this question.

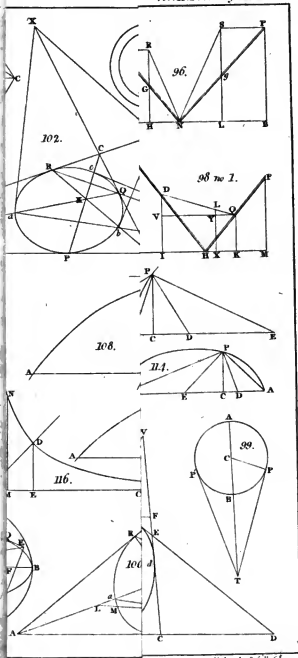
XXIII. QUESTION 203, by Mr. CUNLIFFE.

To find three numbers such, that the sum of every two
shall be a biquadratic, and the sum of all the three a square
number?

SOLUTION, by Mr. CUNLIFFE, the Proposer.

Let x, y , and z , denote the three numbers, and by the ques-
tion, put $x + y = a^4$, $x + z = b^4$, and $y + z = c^4$; half
the sum of these equations is $x + y + z = \frac{1}{2}(a^4 + b^4 + c^4)$
which is to be a square by the question. Let each of the assum-
ed equations be subtracted from the preceding, and there will
be had

$x = \frac{1}{2}(a^4 + b^4 - c^4)$, $y = \frac{1}{2}(a^4 + c^4 - b^4)$, and $z = \frac{1}{2}(c^4 + b^4 - a^4)$;
whence it appears, that the question is reduced to finding three
biquadratic numbers, half the sum of which shall be a square.
And that the required numbers may be all positive, it is neces-
sary that the sum of any two of the three biquadratic numbers
must be greater than the remaining one. Now $m + n, m$ and n ,
are the roots of three biquadratic numbers, half the sum of which
will be a square, viz. $(m^2 + mn + n^2)^2$. But the sum of the
two last, viz. $m^4 + n^4$, is manifestly less than the remaining
one



one, viz. $(m+n)^4 = m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4$, and therefore positive numbers to answer the conditions of the question cannot from thence be had.

In order to obtain positive numbers, put $m+n+sv = a$, $m+rv = b$, and $n+v(r+s) = c$: then

$$\begin{aligned} x+y+z &= \frac{1}{2}(a^4+b^4+c^4) = \frac{1}{2} \{ (m+n+sv)^4 + (m+rv)^4 + (n+v(r+s))^4 \} \\ &= (m^2+mn+n^2)^2 + 2v \times \{ (m+n)^2 \times s + m^2r + n^2 \times (r+s) \} \\ &+ 3v^2 \times \{ (m+n)^2 \times s^2 + m^2r^2 + n^2 \times (r+s)^2 \} \\ &+ 2v^3 \times \{ (m+n) \times s^3 + mr^2 + n \times (r+s)^2 \} + v^4 \times (r^2+rs+s^2)^2, \end{aligned}$$

which is to be a square by the question. In order to abridge the calculation, put $m^2+mn+n^2 = A$; $(m+n)^2 \times s + m^2r + n^2 \times (r+s) = B$; $(m+n)^2 \times s^2 + m^2r^2 + n^2 \times (r+s)^2 = C$; $(m+n) \times s^3 + mr^2 + n \times (r+s)^2 = D$; and $r^2+rs+s^2 = A'$; by means whereof, the foregoing expression to be made a square becomes

$$A^2 + 2Bv + 3Cv^2 + 2Dv^3 + A'^2v^4.$$

Assume $A + (B \div A)v + A'v^2$ for its root, that is, put $A^2 + 2Bv + 3Cv^2 + 2Dv^3 + A'^2v^4 = (A + (B \div A)v + A'v^2)^2 = A^2 + 2Bv + (B^2 \div A^2)v^2 + 2AA'v^2 + (2A'B \div A)v^3 + A'^2v^4$

which gives $v = \frac{B^2 + A^3A' - 3A^2C}{2 \times (A^2D - AA'B)}$

And by writing $-A'$ for A' , we shall have

$$v = \frac{B^2 - A^3A' - 3A^2C}{2 \times (A^2D + AA'B)},$$

which is another value for v .

Two other values for v might be found by assuming $A'v^2 + (D \div A')v \pm A$ for the root of the preceding general expression to be made a square.

A solution was also received from Dr. O'Riordan.

XXIV. QUESTION 204, by Mr. LOWRY.

Of all semi-parabolas having the same area, required that along which a heavy body will descend, by the force of gravity, in the least time possible?

SOLUTION, by Mr. LOWRY, the Proposer.

1. To find a general expression for the time of descent through an arc AB, (fig. 117, pl. 6,) of a parabola having its axis in the vertical line CB. Put $BC = a$, the height CD fallen through in

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the

the time $t = x$, $DE = y$, the parameter $= 4p$, the arch $AE = s$, the velocity acquired in descending through the arch AE , or through the height $CD = v$, and $g = 16\frac{1}{2}$ feet the space fallen through from rest in one second.

Then $\dot{s} \div \dot{t} = v = 2\sqrt{gx}$, or $\dot{t} = \dot{s} \div 2\sqrt{gx}$. But by the nature of the curve, $y^2 = 4p(a-x)$, or $y = 2\sqrt{p(a-x)}$, and $\dot{y} = -\dot{x}\sqrt{p} \div \sqrt{a-x}$; therefore $\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2)} = \dot{x}\sqrt{(1 + p \div (a-x))} = \dot{x}\sqrt{(p+a-x) \div (a-x)}$, and

$$\dot{t} = \frac{\dot{x}}{2\sqrt{gx}} \times \sqrt{\frac{(p+a-x)}{a-x}}.$$

Put $x = az^2$, then $\dot{x} = 2az\dot{z}$, and $\dot{x} \div 2\sqrt{x} = a^{\frac{1}{2}}\dot{z}$, there-

fore, by substitution, $\dot{t} = \frac{a^{\frac{1}{2}}\dot{z}}{g^{\frac{1}{2}}} \sqrt{\frac{p+a-az^2}{a-az^2}} = \frac{\dot{z}}{g^{\frac{1}{2}}} \times$

$\sqrt{\frac{p+a-az^2}{1-z^2}} = \sqrt{\frac{p+a}{g}} \times \dot{z} \sqrt{\frac{1-\frac{az^2}{p+a}}{1-z^2}}$, the flu-

ent to be taken between the limits $z = 0$, and $z = 1$.

Let $e^2 = a \div (p+a)$, and E = the quadrant of an ellipse, of which e is the excentricity, and 1 the semi-transverse axis;

then E is $= \int \dot{z} \sqrt{\frac{1-\frac{az^2}{p+a}}{1-z^2}}$, taken from $z = 0$, to $z = 1$,

and $t = \sqrt{(p+a) \div a} \times E$, the time of descent through the arc AB .

2. We have now to determine the particular values of p and a , so that the area ACB may be a constant quantity, and the time of descent through the arch AB a minimum, or, which is the same thing, that the time may be a constant quantity and the area ACB a minimum.

Now t is $= \sqrt{\frac{p+a}{a}} \times E = \sqrt{\frac{p+a}{a}} \times \int \dot{z} \sqrt{\frac{1-e^2z^2}{1-z^2}}$

and the area $ACB = \frac{1}{2} p^{\frac{1}{2}} a^{\frac{3}{2}}$;

and taking the fluxion of the first equation, we have $(\dot{p} + \dot{a}) E \div 2\sqrt{(p+a)} + E \times \sqrt{(p+a)} = 0$. But to find $\dot{E}$, or the fluxion of the integral $\int \dot{z} \sqrt{((1-e^2z^2) \div (1-z^2))}$, where e and z are variable, we must have recourse to the method of differencing *de curva in curvam* invented by Leibnitz. This method consists in finding the fluxion of such expressions as $\int P\dot{z}$, where P is a function of e and z , the quantity e denoting the parameter,

parameter, excentricity, &c. of a curve; and which is supposed to be constant in the course of the same curve, but to vary in passing from that curve to another which is immediately contiguous. The general expression for the fluxion of $\int P\dot{z}$, when

found by this method, is $P\dot{z} + \dot{e} \int \dot{z} \left(\frac{\dot{P}}{e} \right)$, where $\dot{P}$ is taken on the supposition that e only is variable.

In the above expression, P is $= \sqrt{\frac{1 - e^2 z^2}{1 - z^2}}$, $P\dot{z} = \dot{z} \times \sqrt{\frac{1 - e^2 z^2}{1 - z^2}}$, and $\dot{P} = - \frac{e\dot{e}z^2}{\sqrt{(1 - z^2)(1 - e^2 z^2)}}$; therefore $\dot{e} \int \dot{z} \left(\frac{\dot{P}}{e} \right) = - \dot{e} \int \frac{z^2 \dot{z}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} = - \dot{e} \dot{e} \times \int \frac{z^2 \dot{z}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}}$, and $\dot{E} = \dot{z} \sqrt{\frac{1 - e^2 z^2}{1 - z^2}} - \dot{e} \dot{e} \times \int \frac{z^2 \dot{z}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}}$; but since E is the quadrant of the ellipse, or the whole fluent of $\dot{z} \sqrt{\frac{1 - e^2 z^2}{1 - z^2}}$ from $z = 0$, to $z = 1$, it is evident that the first part of the expression for $\dot{E}$ is $= 0$, and therefore $\dot{E}$ is $= - \dot{e} \dot{e} \int \frac{z^2 \dot{z}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}}$, (taken from $z = 0$, to $z = 1$).

Therefore $\frac{\dot{p} + \dot{a}}{2\sqrt{(p+a)}} \times E - \sqrt{(p+a)} \dot{e} \dot{e} \int \frac{z^2 \dot{z}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} = 0$,
or $(\dot{p} + \dot{a}) \times E - 2(p+a) \dot{e} \dot{e} \int \frac{z^2 \dot{z}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} = 0$.

And taking the fluxion of the second equation, we have

$(\dot{p} \div 2p^{\frac{1}{2}}) \cdot a^{\frac{3}{2}} + \frac{3}{2} a^{\frac{1}{2}} p^{\frac{1}{2}} \dot{a} = 0$, or $a\dot{p} = -3p\dot{a}$, and $\dot{p} = -(3p \div a)\dot{a}$; therefore $\dot{p} + \dot{a} = (1 - 3p \div a)\dot{a}$. But since e^2 is $= a \div (a + p)$, $1 \div e^2$ is $= 1 + p \div a$, or $4 - 3 \div e^2 = 1 - 3p \div a$, and $\dot{p} + \dot{a} = (4 - 3 \div e^2) \dot{a}$. Also $1 - e^2 = p \div (a + p)$, and $\dot{e} \dot{e} = (\dot{a}p - a\dot{p}) \div 2(a + p)^2 = 2\dot{a}p \div (a + p)^2$, therefore $2(p + a) \dot{e} \dot{e} = 4p\dot{a} \div (a + p) = 4(1 - e^2) \dot{a}$, and by substitution,

$$(4 - \frac{3}{e^2}) \dot{a} \times E - 4(1 - e^2) \dot{a} \times \int \frac{z^{\frac{3}{2}}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} = 0,$$

$$\text{or } (4e^2 - 3) E - 4(1 - e^2) \times \int \frac{e^2 z^{\frac{3}{2}}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} = 0.$$

$$\text{But } \frac{e^2 z^{\frac{3}{2}}}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} = \frac{z}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} - z \sqrt{\frac{1 - e^2 z^2}{1 - z^2}},$$

$$\text{therefore } (4e^2 - 3) E - 4(1 - e^2) \times \int \frac{z}{\sqrt{(1 - z^2)(1 - e^2 z^2)}} + 4(1 - e^2) E = 0,$$

$$\text{or } E = 4(1 - e^2) \times \int \frac{z}{\sqrt{(1 - z^2)(1 - e^2 z^2)}}.$$

$$\text{That is, in series, } 1 - \frac{1}{2} e^2 - \frac{1}{2} \cdot \frac{3}{4} e^4 - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} e^6 - \&c.$$

$$= 4(1 - e^2) (1 + \frac{1}{2} e^2 + \frac{1}{2} \cdot \frac{3}{4} e^4 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} e^6 + \&c.)$$

$$\text{or } 3 = (3 - \frac{1}{2} e^2) e^2 + \frac{1}{2} (\frac{7}{2} - \frac{3}{4}) e^4 + \frac{1}{2} \cdot \frac{3}{4} (\frac{11}{3} - \frac{5}{6}) e^6 + \&c.$$

$$\text{or } 3 = \frac{11}{2} e^2 + \frac{1}{2} \cdot \frac{25}{4} e^4 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{39}{6} e^6 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{53}{8} e^8 + \&c.$$

And by reverting the series, we may determine the value of e , and then a and p become known, when either the time or area is given.

The series for e may be found rather differently as follows: Let n be = 3.14159, &c.

$$\text{Then } E \text{ is } = \frac{n}{2} (1 - \frac{1}{2} e^2 - \frac{1}{2} \cdot \frac{3}{4} e^4 - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} e^6 - \&c.)$$

$$= \frac{n}{2} (1 - A e^2 - B e^4 - C e^6 - \&c.)$$

$$\text{and } t = \sqrt{((a + p) \div g)} \times \frac{1}{2} n (1 - A e^2 - B e^4 - C e^6 - \&c.)$$

Taking the fluxion, we have

$$0 = ((\dot{p} + \dot{a}) \div 2\sqrt{(p + a)}) \times (1 - A e^2 - B e^4 - C e^6 - \&c.) + \sqrt{(p + a)} e \dot{e} \times (-2A - 4B e^2 - 6C e^4 - 8D e^6 - \&c.)$$

and substituting for $\dot{p} + \dot{a}$, and $e \dot{e}$, their values, as found above, we have

$$0 = (4e^2 - 3) (1 - A e^2 - B e^4 - C e^6 - \&c.) + 4(1 - e^2) (-2A e^2 - 4B e^4 - 6C e^6 - \&c.)$$

$$\text{or } 3 = \left\{ \begin{array}{l} + \frac{4}{3} A \\ - \frac{8}{3} A \end{array} \right\} e^2 + \left\{ \begin{array}{l} + \frac{8}{4} A \\ - \frac{3}{16} B \end{array} \right\} e^4 + \left\{ \begin{array}{l} + \frac{16}{4} B \\ - \frac{3}{24} C \end{array} \right\} e^6 + \&c.$$

$$\text{That is, } 3 = (4 - 5A) e^2 + (4A - 13B) e^4 + (12B - 21C) e^6 +$$

+

$$+ \&c. \text{ or } 3 = \frac{11}{2^1} e^2 + \frac{1}{2^1} \cdot \frac{25}{4^1} e^4 + \frac{1}{2^1} \cdot \frac{3^2}{4^1} \cdot \frac{39}{6^1} e^6 + \&c.$$

the same as before.

The problem may be resolved in a similar way, when the arch AP, or any function of a and p is given; and the process will not be materially different whatever the nature of the curve may be. In a circle, for example, if the length of the arch be given, we may find the radius so that the time of descent through the given arc, shall be a minimum. Or, which is the same thing, if a pendulum vibrates through a circular arch of a given length, we may find the length of the pendulum, so that the vibrations may be made in the least time.

Put r for the radius CD, (fig. 118, pl. 6,) or the length of the pendulum, BD = half the given arch, $a = DE$, and $x = DI$; then the time of descent through BD, is equal to the

$$\int \frac{r \dot{x}}{2 \sqrt{(2rx - x^2)(a - x)}}, \text{ between the limits } x = a, \text{ and } x = 0;$$

$$\text{and the arch BD} = \int \frac{r \dot{x}}{\sqrt{(2rx - x^2)}} \text{ between the same limits.}$$

By making the fluxions of these expressions = to 0, we may after proper reduction, obtain the value of r in a series;

$$\text{or, since } t = \frac{n}{2} \sqrt{\frac{r}{2g}} \times \left\{ 1 + \left(\frac{1}{2}\right)^2 \cdot \frac{a}{2r} + \left(\frac{1}{2} \cdot \frac{1}{2}\right)^2 \cdot \frac{a^2}{2^2 r^2} + \left(\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\right)^2 \cdot \frac{a^3}{2^3 r^3} + \&c \right\}$$

$$\text{and the arch DB} = \sqrt{(2ra)} \times \left(1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{a}{2r} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{a^2}{2^2 r^2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{a^3}{2^3 r^3} + \&c. \right)$$

we may find r by taking the fluxions of these series and exterminating a , as we have done in the example above, but the process will be a little tedious.

XXV. QUESTION 205, *by* URSA MINOR.

Suppose a right cone to be cut by a plane, and the curved surface of the part cut off to be extended into a plane surface. Required the nature of the plane curve, formed by the common section of the cone and cutting plane?

SOLUTION, by Mr. WALLACE, R. M. College.

Let V, (fig. 119, pl. 6,) be the vertex of the cone, APB its section by the plane, VC its axis meeting the section in C, AVB a section of the cone through its axis, and perpendicular to the section APB, ACB the common section of the two planes AVB and APB. Draw VP in the surface of the cone to P any point in the curve APB, and join CP. Let EQF be a section of the cone, perpendicular to its axis, (which section will of course be a circle,) and

and let FV (the distance of its circumference from the vertex,) be unity. Let this circle meet VB, VP in F and Q, and draw FD, QD to its centre. Put $VC = d$, the angle CVB or CVP = α , the angle BCV = β . Also put the indeterminate line VP = r , and the indeterminate arch FQ = ϕ . Then because FD = $\sin \alpha$, the angle FDQ, (that is the angle contained by the planes BCV, PCV) will be $\phi \div \sin \alpha = n\phi$, (putting $n = 1 \div \sin \phi$).

Now, considering the three planes VCB, BCP, VCP as forming a solid angle at C, and that the plane VCB is perpendicular to BCP, we have by spherics, $\tan VCP = \tan VCB \div \cos FDQ = \tan \beta \div \cos n\phi$. Draw PG perpendicular to VC; then in the right-angled triangle VPG, we have $PG = VP \times \sin PVG = r \sin \alpha$, and $VG = VP \times \cos PVG = r \cos \alpha$. And in the right-angled triangle PGC, we have $CG = PG \times (1 \div \tan VCP) = r \sin \alpha \cos n\phi \div \tan \beta$. But $VG + GC = VC = d$; therefore $r \cos \alpha + r \sin \alpha \cos n\phi \div \tan \beta = d$, and hence

$$r = \frac{\tan \beta}{\cos \alpha \tan \beta + \sin \alpha \cos n\phi} \times d,$$

or putting $\sin \beta \div \cos \beta$ instead of $\tan \beta$,

$$r = \frac{\sin \beta}{\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos n\phi} \times d.$$

Conceive now the surface of the cone to be cut along the line VA, and extended into a plane, as in fig. 120, pl. 6. Then the surface will form the figure AVAPB, and the circle EFQ in fig. 119, will become an arc EFQE, in fig. 120, having its radius = 1. Therefore in the plane figure AVA, fig. 120, the nature of the curve BPA, is defined by the equation

$$r = \frac{\sin \beta}{\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos n\phi} \times d,$$

where α and β denote given angles, d a given line, and where ϕ denotes the indeterminate angle BVP, (VB being supposed given by position), and r the indeterminate line VP.

Because $d = (\sin (\alpha + \beta) \div \sin \beta) \times VB$, the equation may be otherwise expressed thus

$$r = \frac{\sin (\alpha + \beta)}{\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos n\phi} \times VB,$$

which again may be transformed to

$$r = \frac{\tan \beta + \tan \alpha}{\tan \beta + \tan \alpha \cos \phi} \times VB.$$

If the section be a parabola, then $\phi\alpha = \beta$, and the equation becomes simply,

$$r = \frac{2}{1 + \cos n\phi} \times VB.$$

XXVI. QUESTION 206, *by Mr. P. R.*

Two magnetical poles being given in position ; the force of each of which is supposed to be as the m th. power of the distance from it reciprocally ; it is required to find a curve, in any point of which a needle (indefinitely short) being placed, its direction, when at rest, may be a tangent to the curve ?

The following SOLUTION to this question, is taken from the Supplement to the Encyclopedia Britannica, Art. Magnetism, written by Dr. Robison, who there informs us that he received it from Mr. Playfair, of the University of Edinburgh.

1. Let A and B, (fig. 121, pl. 6,) be the poles of a magnet, C any point in the curve required ; then we may suppose the one of these poles to act on the needle only by repulsion, and the other only by attraction, and the direction of the needle, when at rest, will be the diagonal of a parallelogram, the sides of which represent these forces. Therefore, having joined AC and BC,

let AD be drawn parallel to BC, and make $\frac{1}{AC^m} : \frac{1}{BC^m} :: AC :$

AD ; join CD, then CDF will touch the curve in C.

2. Hence an expression for AF may be obtained. For, by the construction, $AD = \frac{AC^{m+1}}{BC^m}$, and since $BC : AD ::$

$BF : FA$, and $BC - AD : AD :: AB : AF$, we have $AF = \frac{AB \times AC^{m+1}}{BC^{m+1} - AC^{m+1}}$.

3. A fluxionary expression for AF may also be found in terms of the angles CAB, ABC. In CF take the indefinitely small part CH, draw AH, BH, and from C draw CL perpendicular to AH and CK to BH: Draw also BG and AM at right angles to FH. Let the angles $CAB = \phi$, and $CBA = \psi$; then $CAH = \dot{\phi}$, and $CBH = -\dot{\psi}$; also, $CL = AC \times \dot{\phi}$, and $CK = -BC \times \dot{\psi}$. Now $HC : CL :: AC : AM = \frac{AC^2 \times \dot{\phi}}{HC}$;

and for the same reason, $BG = -\frac{BC^2 \times \dot{\psi}}{HC}$. Therefore,

since $AF : FB :: AM : BG$, $AF : FB :: \frac{AC^2 \times \dot{\phi}}{HC} : -\frac{BC^2 \times \dot{\psi}}{HC}$, and $AF : AB :: \sin \psi^2 \dot{\phi} : -\sin \psi^2 \dot{\phi} - \sin \phi^2 \dot{\psi}$;

wherefore, if $AB = a$, $AF = \frac{-a \sin \psi^2 \dot{\phi}}{\dot{\psi} \sin \phi^2 + \dot{\phi} \sin \psi^2}$.

4. If this value of AF be put equal to that already found, a fluxionary equation will be obtained, by the integration of which

the curve may be constructed. Because $AF = \frac{AB \times AC^{m+1}}{BC^{m+1} - AC^{m+1}}$

and since $AC = \frac{a \sin \psi}{\sin (\phi + \psi)}$, and $BC = \frac{a \sin \phi}{\sin (\phi + \psi)}$,

we have, by substitution, $AF = \frac{a \sin \psi^{m+1}}{\sin \phi^{m+1} - \sin \psi^{m+1}} =$

$-\frac{a \dot{\phi} \sin \psi^2}{\dot{\psi} \sin \phi^2 + \dot{\phi} \sin \psi^2}$. Hence, $\sin \phi^2 \times \dot{\psi} \sin \psi^{m+1} +$

$\dot{\phi} \sin \psi^{m+3} = -\sin \psi^2 \times \dot{\phi} \sin \phi^{m+1} + \dot{\phi} \sin \psi^{m+3}$, and

therefore, $\dot{\psi} \sin \psi^{m-1} = -\dot{\phi} \sin \phi^{m-1}$; and also

$\int \dot{\psi} \sin \psi^{m-1} + \int \dot{\phi} \sin \phi^{m-1} = C$.

5. These fluents are easily found, when m is any whole positive number.

If $m=1$, we have $\dot{\psi} + \dot{\phi} = 0$; and $\phi + \psi = C$,

$m=2$, we have $\dot{\psi} \sin \psi + \dot{\phi} \sin \phi = 0$; and $\cos \phi + \cos \psi = C$; &c.

The first of above equations belongs to a segment of a circle described upon AB , which therefore would be the curve required if the magnetical force were inversely as the distances.

If the magnetical force be inversely as the square of the distance, that is, if $m=2$, $\cos \phi + \cos \psi$ is equal to a constant quantity. Hence, if, beside the points A and B , any other point be given in the curve, the whole may be described. For instance, let the point E , (fig. 122, pl. 6,) be given in the curve, and in the line DE which bisects AB at right angles. Describe from the centre A , a circle through E , viz. QER ; then AD being the cosine of DAE to the radius AE , the sum of the cosines of ϕ and ψ will be every where (to the same radius) $= 2AD = AB$. Therefore to find E' , the point in which any other line AN , making a given angle with AB , meets the curve, draw from N ,

N, the point in which it meets the circumference of the circle QER, NO, perpendicular to AB, so that AO may be the cosine of NAO, and from O toward A take $OP = AB$, then AP will be the cosine of the angle ABE'; so to find BE', draw PQ perpendicular to AP, meeting the circle in Q; join AQ, and draw BE' parallel to AQ, meeting AE' in E', the point E' is in the curve. In this way the other points of the curve may be found.

XXVII. QUESTION 207, by Mr. WALLACE, R. M. College.

Shew that the rectification of the curve called the *line of sines* depends upon that of an ellipse, without employing the fluxionary calculus, or in any other way comparing the indefinitely little increments of the curves?

SOLUTION, by Mr. WALLACE, the Proposer.

Let AEB, (fig. 123, pl. 6,) be half the base of a cylinder, and AFB a section of the cylinder through C the centre of its base, making with it an angle of 45° , and which will be a semi-ellipse: Let CEF be a plane perpendicular to AB. From P, any point in APEB, the semi-circumference of the base, draw PQ in the surface of the cylinder perpendicular to the base, and meeting the ellipse AFB in Q. Through PQ, draw a plane perpendicular to AB, meeting the base in PD, and the plane AFB in QD. Then it is easy to see that $DP = PQ$, and consequently, that PQ is the sine of the arch AP. Hence it follows, that the portion of the cylindric surface contained between the semi-circle AEB, and semi-ellipse AFB would, if expanded into a plane, form a figure of sines, having for its base AEB, the semi-circumference of the of the cylinder. Now if it be remarked that $CF : CA :: \sqrt{2} : 1$; it will be evident from what has been said, that the length of the line of sines is equal to half the perimeter of an ellipse whose axes are to one another as $\sqrt{2}$ to 1, and whose conjugate axis is equal to the diameter of the circle from which the line of sines is conceived to be generated.

From what has been shewn, it is also easy to see that we can always express any arch of the line of sines by a corresponding arch of the same ellipse.

XXVIII. QUESTION 208, by Mr. IVORY.

Two points being given in a vertical plane, but not in the same horizontal line; it is required to determine the position of three inclined planes, such, that (supposing the body to pass from one plane to another, without any loss of velocity,) the time of descent

cent may be less than the time on any other three planes, when the line of descent is of a given length ?

SOLUTION, by Mr. IVORY, the Proposer.

As the solution of this question seems to require complicated calculations, I shall be content with pointing out the method to be pursued. For the greater simplicity, the number of planes may be supposed to be two in place of three: then, preserving

the notation in question 188, the time of the fall is $\frac{\epsilon'}{a-x} \times$

$(\sqrt{a-x}) + \frac{\epsilon}{x} \sqrt{x}$, which is to be a minimum; and the

line of descent, is $\epsilon + \epsilon'$, which is to be of a given length, or equal to a given quantity, A . Let m be an indeterminate quantity, and having multiplied $\epsilon + \epsilon'$ by m , let the product be added to the function which is to be a minimum, and denote the sum

by $f(x, y)$; that is, let $f(x, y) = \frac{\epsilon'}{a-x} \times (\sqrt{a-x})$

$+ m\epsilon + \frac{\epsilon}{x} \sqrt{x} + m\epsilon$. The co-ordinates x and y , are next

to be determined so as to make the function $f(x, y)$ a mini-

mum: and, for this end, the equations $\left(\frac{df(x, y)}{dx}\right) = 0$, and

$\left(\frac{df(x, y)}{dy}\right) = 0$, must be resolved, (see question 188). The

values of x and y , thus found, besides involving the quantities given in the problem, will likewise contain the indeterminate quantity m : and being substituted in the equation $\epsilon + \epsilon' = A$, or $\sqrt{(x^2 + y^2)} + \sqrt{((a-x)^2 + (b-y)^2)} = A$, there will be obtained an equation which must be satisfied by taking m of a proper value.

If there be three inclined planes, the function which is to be made a minimum, will contain four variable co-ordinates; and a like number of equations for determining these will be obtained by taking the partial fluxions in that function; in other respects the same method of solution is to be pursued as shewn in the case of two inclined planes.

XXIX. QUESTION 209, by CURIOSUS.

Find the shortest distance between two given points on the surface of a spheroid ?

To this question we have received no answer.

XXX

XXX. PRIZE QUESTION 210, by Mr. LOWRY.

In a given ellipse to inscribe a polygon of a given number of sides, so that each side may pass through a given point?

SOLUTION, by Mr. LOWRY, the Proposer.

LEMMA. Let a, b , (fig. 124, pl. 6.) be two points in an ellipse, and let a circle be described on the conjugate axis XY ; draw aI, bK parallel to the transverse axis meeting the circle in A and B , and XY in I and K . Join ab, AB , and from any point p , in ab , draw pm parallel to aI or bK , meeting AB in P , and XY in m ; then pm , is to Pm , as the transverse axis to the conjugate axis.

For $aI^2 : bK^2 :: XI.IY = IA^2 : XK.KY = KB^2$;
therefore $aI : bK :: IA : KB$,

or $aI : IA :: bK : KB :: pm : Pm$;
therefore $pm^2 : Pm^2 :: aI^2 : IA^2 = XI.IY ::$ square of the transverse : the square of the conjugate, or pm is to Pm , as the transverse axis to the conjugate axis.

Now let $abcde$, &c. (fig. 125, pl. 6,) be the polygon inscribed in the given ellipse, so that the sides ab, bc, cd, de , &c. may pass through the given points p, q, r, s, t , &c. respectively; and let a circle be described on the conjugate axis XY . Draw aA, bB, cC, dD, eE , &c. parallel to the transverse axis of the ellipse meeting the circle in the points A, B, C, D, E , &c. and join AB, BC, CD, DE, EF , &c. Draw pP, qQ, rR, sS, tT , &c. from the given points p, q, r, s, t , &c. parallel to the transverse axis, meeting the lines AB, BC, CD, DE, EF , &c. in P, Q, R, S, T , &c. and produce pP to meet XY in m . Then by the lemma, pm is to Pm , as the transverse axis to the conjugate axis; and since p is a given point, the distance pm is given; therefore Pm is given, and P is a given point. And in a similar way it is shewn that Q, R, S, T , &c. are given points. Wherefore it is evident that if a polygon $ABCDEF$, &c. be inscribed in the circle, so that its sides AB, BC, CD, DE, EF , &c. may pass through the given points P, Q, R, S, T , &c. the points a, b, c, d, e , &c. in the ellipse, may be found by drawing lines from the angular points of the polygon inscribed in the circle, parallel to the transverse axis of the ellipse.

To inscribe a polygon in a circle so that its sides shall pass through given points, is a problem that has been often resolved; and the solution that I shall give here is not materially different from others that have appeared before, but I shall endeavour to put it in rather a simpler point of view.

And first to inscribe a triangle ABC (fig. 126, 127, pl. 6,) in a given circle, so that its sides AB, BC, AC shall pass through the given points P, Q, R .

Join

Join PQ , and make the rectangle HQP equal to the given rectangle CQB , and draw HC to meet the circle in K and join AK ; then H is a given point, and the points B, C, H, P are in the circumference of the same circle; therefore the angle QHC is $= CBP = AKH$ in fig. 126, or its supplement in fig. 127 (22.3), and AK is parallel to PQ . To the centre O , draw HO meeting the circle in N , and AK in I , and join CN . Make the rectangle $HO.OP' =$ to the square of the semi-diameter of the circle, and draw CP' to meet the circle in G , and join KG ; then P' is a given point, and (Playfair's Euclid VI. P. F. Cor.) CN bisects the angle GCK ; therefore the arcs KN, NG are equal, and consequently KG is perpendicular to HO : therefore the angle ACP' , or IKG , is the complement of the angle KIH , or equal to the difference between the given angle PHO and a right angle. Hence we have this

CONSTRUCTION: Find the points H and P' as in the analysis, and on $P'R$ describe a segment of a circle to contain an angle equal to the difference between the angle PHO and a right angle, and let it meet the given circle in C ; draw QCB, CRA to meet the circle in B and A , then the line which joins the points B, A will pass through the given point P .

DEMONSTRATION. Draw CH, CP' to meet the circle in K and G , and join AK, KG and CN ; then because $HO.OP' = ON^2$, KG is perpendicular to OH as is shewn in the analysis, and therefore the angle IKG or ACP' is the complement of KIH . But by construction the angle ACP' is the complement of PHO , therefore the angle HIK is equal to the angle PHO , and AK is parallel to PQ ; therefore the angle QHC is $= HKI = CBA$. But because the rectangle HQP is $=$ to the rectangle CQB , the points B, C, H, P are in the circumference of the same circle, therefore the angle QHC is $= CBP$, and consequently the angle CBA is $=$ to the angle CBP ; therefore the line BA passes through P .

The solution of this particular case, immediately leads to the solution of the general problem, when the polygon has any number of sides whatever.

Let the points HP' (fig. 128, pl. 6,) be found as before for the given points P, Q ; and join $P'R$, (the point R being that through which the side CD is to pass). On $P'R$, produced if necessary, find the point H' , so that the rectangle $P'R.H'$ may be equal to the given rectangle DRC , and produce OH' to P'' , so that the rectangle $P'.OH'$ may be equal to the square of the semi-diameter, and join $P'D$ meeting the circle in m ; then H' and P'' are given points, and because the construction for the points H, P' as related to the points P, R ; is the same as the construction for the points H, P' as related to the points P, Q ; it is evident from the preceding analysis that the angle mDG is the difference

difference between the angle $P'H'P$ and a right angle, and is therefore given; and it is shewn in the preceding case, that the angle ACG or ADG is given; therefore the angle ADm , or or ADP'' is given. Hence, when the number of sides of the polygon is four, or AD passes through a given point S' , a circle described on $P'S'$, to contain an angle equal to the given one, will intersect the circle in D , and the construction will then be obvious.

If the polygon has more sides than four, join $P'S$ (S being the point through which the side DE is to pass), and find the point H'' in $P'S$, so that the rectangle $P'SH''$ may be equal to the given rectangle DSE , and in $H''O$ produced take the point P'' , so that the rectangle $H''OP''$ may be equal to the square of the semi-diameter, and join $P''E$ meeting the circle in n : then H'' and P'' are given points, and by the foregoing analysis the angle $P''Em$ is given, being equal to the difference between the angle $OH''P''$ and a right angle; therefore the angle mEn is given, and the angle AEm or ADP has been shewn to be given; therefore the angle AEn is given: and, if AE be the last side of the polygon, a circle described through the points $P''T$ to contain the given angle will intersect the given circle in E . And in the same manner we may proceed for any polygon whatever, by joining the point last found, and the point through which the next side of the polygon is to pass, and making a similar construction to that which we have made above.

Mr. LOWRY is requested to send to Mr. GLENDINNING's for the Medal for solving the Prize Question.



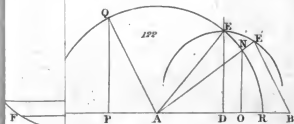
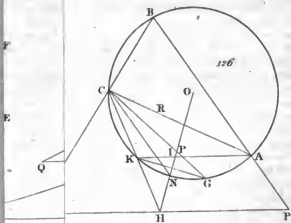
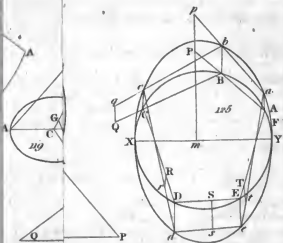
Since the preceding sheets were printed off, we have received the following solution to question XI. (191).

XI. QUESTION 191, *from the* CAMB. UN. CAL.

Bradley observed that every star passed the meridian farthest south when it came about six in the morning, whatever were its position with respect to the cardinal points of the ecliptic. Suppose then the place of a star to be given; on what day will it pass the meridian of London farthest to the south; and at what hour will it pass it on that day?

SOLUTION, *by* ASTRONOMICUS.

The greatest apparent distance of a star from the north pole, arising from aberration and nutation together, is when the sun's longitude, and also that of the moon's ascending node, are three signs *before* the star, reckoning according to the order of the signs; and that apparent distance will be greatest of all when the star is in the solstitial colure; but it does not appear, that the period can be accurately determined when such an event will take place, for any proposed star. We can, however, calculate the time which seems to be required by the question, if it falls within the limits of some proposed revolution of the moon's nodes, or if it is restricted to a given year: Thus, for example, let the year be 1809, and suppose the star's right ascension = 108° , and its declination = 60° north. Let a great circle be drawn, (to the eastward) from the star, perpendicular to the hour circle or meridian passing through the star, and it will meet the ecliptic in $25^{\circ} 41'$ of *libra*, which is the sun's place, when the aberration in declination southward is a *maximum*, (the obliquity of the ecliptic being $23^{\circ} 27' 44''$.) this answers to Oct. 19, at London; on that day the star comes to the meridian at $22\frac{1}{2}$ min. before 6 in the morning. The longitude of the moon's node at that time is $6s. 23^{\circ} 45'$, and that of the sun $6s. 25^{\circ} 26'$; the star therefore, is very nearly at its maximum distance from the north pole, which distance will not sensibly vary during 3 or 4 days at that time.





NOTICES

RELATING TO MATHEMATICS.

I. MATHEMATICIANS LATELY DECEASED.



On the 7th of April, 1807, M. D. Lalande, the celebrated French Astronomer, aged 75. He was the Author of many Memoirs connected with his favourite Science. His most valuable and best known work, however, is his *Astronomie* in 3 vols. Quarto, of which the third edition was published in 1792.

In July, 1807, George Atwood, Esq. M. A. and F. R. S. aged 62. He was for some time Tutor, and many years Fellow of Trinity College, Cambridge; and was highly distinguished for his Mathematical Acquirements. In 1784, he published in one volume, octavo, "A Treatise on the Rectilinear Motion, and Rotation of Bodies, with a description of original Experiments relative to the subject." At the same time he published, "An Analysis of a Course of Lectures, on the Principles of Natural Philosophy, read in the University of Cambridge." In 1801, he published a "Dissertation on the Construction and Properties of Arches," and in the course of that year, he also published a Supplement to it. Mr. Atwood also, at different times, contributed several valuable Memoirs to the Transactions of the Royal Society of London.

II. ANOTHER NEW PLANET.

On the 29th of March, 1807, Dr. Olbers, at Bremen, discovered another new planet, to which he has given the name of Vesta, being the second which we owe to the observations of this learned and indefatigable astronomer. It resembles a star of the sixth magnitude, and is very like in its appearance to the Georgium Sidus. This planet is found in the same region between Mars and Jupiter, in which Ceres, Pallas, and Juno, perform their revolutions round the Sun. Its place, as observed by Mr. Firminger at the Royal Observatory, Greenwich, on the 22d of June, 1807, 10^h. 29^m. 48^s. mean time was

App. A. R.	App. Dec. N.	Longitude.	Lat N.
6 ^h . 4 ^m . 23 ^s . 30"	— 6 ^h . 36 ^m . 3 ^s	— 6 ^h . 1 ^m . 23 ^s . 18"	— 7 ^h . 47 ^m . 0 ^s .

III. TRANSACTIONS OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH.

We propose, in future, to give the titles of the papers relating to Mathematics, contained in the Transactions of the Royal Society of London, as they appear. And we shall, at present, enumerate all those which have been published in that work, since the commencement of the present century.

The Vol. for 1800 contains, 1. On the method of determining from the real probabilities of life, the nature of contingent reversions, in which three lives are involved in the survivorship. By Wm. Morgan, Esq. F. R. S. 2. On the power of penetrating into space by telescopes, &c. By Dr. Herschell. 3. A second appendix to the improved solution of a problem in physical astronomy, inserted in the philosophical transactions for 1798. By the Rev. John Hellins, B. D. F. R. S. 4. Outlines of experiments and inquiries respecting sound. By Thomas Young, M. D. F. R. S. 5. On double images, caused by atmospherical refraction. By Wm. Hyde, Wollaston, M. D. F. R. S. 6. Investigation of the powers of the prismatic colours to heat and illuminate objects: with remarks that prove the different refrangibility of radiant heat. By Dr. Herschell. 7. Experiments on the refrangibility of the invisible rays of the sun. By Dr. Herschell. 8. Experiments on the solar and on the terrestrial rays that occasion heat, &c.

By Dr. Herschell. 9. An account of the trigonometrical survey, carried on in the years 1797, 1798, and 1799. By Captain Win. Mudge, of the Royal Artillery.

The Vol. for 1801 contains, 1. The Bakerian Lecture. On the mechanism of the eye. By Dr. Young. 2. On the necessary truth of certain conclusions, obtained by means of imaginary quantities. By Robert Woodhouse, A. M. 3. Demonstration of a theorem by which such portions of the solidity of a sphere are assigned as admit of an algebraic expression. By Mr. Woodhouse. 4. Observations tending to investigate the nature of the sun, &c. By Dr. Herschell. 5. On an improved reflecting circle. By Joseph de Mendoza Rios, Esq. F. R. S.

The Vol. for 1802 contains, 1. The Bakerian Lecture. On the theory of light and colours. By Dr. Young. 2. On the independence of the analytical and geometrical methods of investigation, and on the advantages to be derived from their separation. By Mr. Woodhouse. 3. Observations on the two lately discovered celestial bodies. By Dr. Herschell. 4. A method of examining refracting and dispersive powers, by prismatic reflection. By Dr. Wollaston. 5. On the oblique refraction of Iceland crystal. By Dr. Wollaston. 6. An account of some cases of the production of colours not hitherto described. By Dr. Young. 7. Of the rectification of the conic sections. By the Rev. Mr. Hellins.

The Vol. for 1803 contains, 1. The Bakerian Lecture. Observations on the quantity of horizontal refraction; with a method of measuring the dip at sea. By Dr. Wollaston. 2. Observations on the transit of Mercury over the disk of the sun, &c. By Dr. Herschell. 3. Account of the changes that have happened during the last twenty-five years, in the relative situation of double stars, &c. By Dr. Herschell. 4. Account of the measurement of an arc of the meridian, extending from Dunmose, in the Isle of Wight, to Clifton, in Yorkshire, in course of the operations carried on for the trigonometrical survey of England in the years 1800, 1801, and 1802. By Major Mudge.

The Vol. for 1804 contains, 1. The Bakerian Lecture. Experiments and calculations relative to physical optics. By Dr. Young. 2. On the integration of certain differential expressions, with which problems in physical astronomy are connected, &c. By Mr. Woodhouse. 3. Continuation of an account of the changes that have happened in the relative situation of double stars. By Dr. Herschell.

The Vol. for 1805 contains, 1. Experiments for ascertaining how far telescopes will enable us to determine very small angles, &c. with an application of the result of these experiments to a series of observations on the nature and magnitude of Mr. Harding's lately discovered star. By Dr. Herschell. 2. An essay on the cohesion of fluids. By Dr. Young. 3. On the direction and velocity of the motion of the sun and solar system. By Dr. Herschell. 4. Observations on the singular figure of the planet Saturn. By Dr. Herschell.

The Vol. for 1806 contains, 1. The Bakerian Lecture. On the force of percussion. By Dr. Wollaston. 2. Mémoire sur les quantités imaginaires. Par M. Buée. 3. The application of a method of differences to the species of series whose sums are obtained by Mr. Landen, by the help of impossible quantities. By Mr. Benj. Gompertz. 4. On the quantity and velocity of the solar motion. By Dr. Herschell. 5. A new demonstration of the binomial theorem, when the exponent is a positive or ne-

gative fraction. By the Rev. Abram Robertson, A. M. F. R. S. Savilian Professor of Geometry in the University of Oxford. 6. New method of computing logarithms. By Thomas Manning, Esq. 7. On the declination of some of the principal fixed stars: with a description of an astronomical circle. By John Pond, Esq. 8. Observations and remarks on the figure, the climate, and the atmosphere, of Saturn and its ring. By Dr. Herschell.

The Vol. for 1807 contains, 1. On the precession of the equinoxes. By the Rev. Abram Robertson. 2. An investigation of the general term of an important series in the inverse method of finite differences. By the Rev. John Brinkley, D. D. F. R. S. and Andrew's Professor of Astronomy in the University of Dublin. 3. Experiments for investigating the cause of the coloured concentric rings, discovered by Sir Isaac Newton, between two object glasses laid one upon another. By Dr. Herschell. 4. Observations and measurements of the planet Vesta. By Mr. Schroeter. 5. Observations on the nature of the new celestial body, discovered by Dr. Olbers, and of the comet which was expected to appear last January, on its return from the sun. By Dr. Herschell.

The first part of the sixth vol. of the Transactions of the Royal Society of Edinburgh is now published. As it, perhaps, may be agreeable to our readers to have a list of the memoirs relating to mathematical subjects contained in this work from its commencement, we shall enumerate them as follows.

VOL. I. contains, 1. Biographical account of Dr. Matthew Stewart. By Mr. Professor Playfair. 2. On the causes which affect the accuracy of barometrical measurements. By Mr. Playfair. 3. On the use of negative quantities in the solution of problems. By Mr. Wm. Greenfield. 4. An improvement of the method of correcting the observed distance of the moon from the sun or from a fixed star. By the Rev. Thomas Elliott. 5. Orbit and motion of the Georgium Sidus, determined directly from observations. By Mr. John Robison, Professor of Natural Philosophy in the University of Edinburgh.

VOL. II. contains, 1. Abstract of experiments made to determine the true resistance of the air to the surfaces of bodies. By Dr. Charles Hutton, of the Royal Military Academy, Woolwich. 2. Observations on the places of the Georgium Sidus, made at Edinburgh with an equatorial instrument. By Professor Robison. 3. On the motion of light, as affected by refracting and reflecting substances, which are also in motion. By Professor Robison. 4. Demonstrations of some of Dr. Stewart's general theorems. By the Rev. Dr. Small. 5. Remarks on the astronomy of the Brahmins. By Mr. Playfair. 6. On the resolution of indeterminate problems. By Mr. John Leslie.

VOL. III. contains, 1. Experiments and observations on the unequal refrangibility of light. By Dr. Robert Blair, Professor of Practical Astronomy in the University of Edinburgh. 2. On the origin and investigation of Porisms. By Mr. Playfair.

VOL. IV. contains, 1. Four theorems for resolving all the cases of plane and spherical triangles. By the Rev. Mr. Fisher. 2. On the principles of the antecedental calculus. By James Glenie, Esq. 3. Observations on the trigonometrical tables of the Brahmins. By Mr. Playfair. 4. Some geometrical porisms, with examples of their application to the solution of problems. By Mr. Wm. Wallace. 5. On the latitude and

longitude of Aberdeen. By Dr. Andrew Mackay. 6, A new series for the rectification of the ellipsis, with observations of the evolution of the formula $(a^2 + b^2 - 2ab \cos. \phi)^2$. By James Ivory, A. M.

VOL. V. contains, 1. Investigation of certain theorems relating to the figure of the earth. By Mr. Playfair. 2. New method of resolving cubic equations. By Mr. Ivory. 3. A new and universal solution of Kepler's problem. By Mr. Ivory. 4. A new method of expressing the coefficients of the development of the algebraic formula $(a^2 + b^2 - 2ab \cos. \phi)^2$, by means of the perimeters of two ellipses; with an appendix, containing an investigation of a formula for the rectification of any arch of an ellipse. By Mr. Wallace. 5. Rule for reducing a square root to a continued fraction. By Mr. Ivory.

VOL. VI. PART I. contains a geometrical investigation of some curious and interesting properties of the circle, &c. By Mr. Glenic.

IV. MATHEMATICAL WORKS LATELY PUBLISHED.

Scriptores Logarithmici, Vol. VI. 4to. By Francis Maseres, Esq. F. R. S. *Cursitor Baron of the Exchequer*.

An Elementary Treatise on Natural Philosophy. Translated from the French of M. R. I. Haüy, Professor of Mineralogy at the Museum of Natural History, &c. By Olinthus Gregory, A. M. of the Royal Military Academy, Woolwich. 2 Vols, 8vo. with plates. A second Edition of the original French Work has also been lately published, and Mr. Gregory has availed himself of it in his Translation.

A Reply to a critical and monthly Reviewer, in which is inserted Euler's Demonstration of the Binomial Theorem. By Abram Robertson, D. D. F. R. S. Savilian Professor of Geometry.

FOREIGN BOOKS.

Journal De L'Ecole Polytechnique, 13th Cahier. The 9th and 10th Cahiers, containing the remainder of the *Mechanique Philosophique*, by Prony; and the 14th Cahier are in the Press.

Application De L'Analyse A La Geometrie, par Monge and Hachette, 2 Parts in Quarto.

Correspondence sur Le Ecole Polytechnique, 6 Numeros in Octavo.

Essai De Geometrie Analytique appliqué aux Courbes et aux Surfaces du Second Ordre. Par I. B. Biot. Second Edition.

A French Translation of the *Disquisitiones Arithmeticae* of Mr. Gauss has appeared. It is in 1 Vol. 4to. and is called *Recherches Arithmetiques*. Our readers will recollect that we have already noticed the original work at page 77 of Vol. I of the *Mathematical Repository*, where we have given his formula for the calculation of the side of a regular polygon of 17 sides, inscribed in a circle.

Nouvelle Méthode pour la Résolution des Équations Numériques d'un degré quelconque; D'après laquelle tout le calcul exigé pour cette Résolution se réduit à l'emploi des deux premières règles de l'Arithmétique. Par F. D. Budan.

Mémoire sur la relation qui existe entre les distances de cinq points quelconques pris dans l'espace, avec un Essai sur la Théorie des Transversales. Par M. Carnot.

ARTICLE IV.

*Solutions to Questions proposed in Number VII.*I. QUESTION 211, *from the BIJA GONETA, or HINDOO ALGEBRA.*

Find two integer square numbers whose sum and difference diminished by 1 shall both be squares?

"The *Bija Goneta* was written in Sanscrit by BHASKER ACHARIJ, a famous Hindoo Mathematician and Astronomer, who lived about the beginning of the 13th century of the Christian era: and was translated into Persian in 1634."

FIRST SOLUTION, *from the BIJA GONETA, or HINDOO ALGEBRA.*

Assume $4x^2$ and $5x^2 + 1$ for the two squares:

Then $4x^2 + 5x^2 + 1 - 1 = 9x^2$ a square: And $5x^2 + 1 - 4x^2 - 1 = x^2$ a square:

Therefore we have to make $5x^2 + 1$ a square: Now if we assume $x = 4$, then $5 \times 16 + 1 = 81$, and $4 \times 16 = 64$, are the two squares: For $81 + 64 - 1 = 144$ a square; and $81 - 64 - 1 = 16$ a square. And if $x = 72$, the two squares will be 25921 and 20736.

We shall subjoin the three following Questions, with the methods of solution from the same work, (the *Bija Goneta*.)

1. What two integer numbers are those (5 and 6 excepted,) that when 3 is added to twice the difference of their squares, and also to 3 times that difference, the two results shall be square numbers?

Let x = the difference of the squares. Then $2x + 3 = y^2$ (a square):
And $3x + 3 = z^2$ (a square):

Since $2x + 3 = y^2$, $x = \frac{y^2 - 3}{2}$, and $3x + 3 = \frac{3y^2 - 3}{2} = z^2$,

whence $9y^2 = 6z^2 + 9$, and $3y = (6z^2 + 9)^{\frac{1}{2}}$. If z be assumed $= 6$, then $3y = 15$, and $y = 5$, and $x = 11$, and the two numbers are 5 and 6, which, by the question, are excepted.

Now let $z = 60$; then $3y = 147$, and $y = 49$, whence
VOL. II. PART I. 2 C $x =$

the root $\frac{7y}{3}$ instead of $\frac{5y^2}{3}$ (or xy) in rendering the expression $xy(x+y) + 1$ a rational square: For the same value of y will result if we use $\frac{5y^2}{3}$; thus let $\frac{5y^2}{3} \times \frac{8y}{3} + 1 = b^2$ (a square); then $40y^3 + 9 = 9b^2$; and the expression $40y^3 + 9$ is a square when $y = 1$, or 6.

SECOND SOLUTION, by Mr. CUNLIFFE, R. M. College.

Let $a^2 + 2ab + b^2$ and a^2 denote the two squares: Then by the question

$$2a^2 + 2ab + b^2 - 1 = c^2,$$

$$\text{and} \quad 2ab + b^2 - 1 = d^2.$$

The sum and difference of these equations, give

$$2a^2 + 4ab + 2b^2 - 2 = c^2 + d^2,$$

$$\text{and} \quad 2a^2 = c^2 - d^2.$$

Whence $a^2 + 2ab + b^2 = (a+b)^2 = \frac{1}{2}(c^2 + d^2) + 1$, and $a^2 = \frac{1}{2}(c^2 - d^2)$.

Put $d + n = c$; then $a^2 = \frac{1}{2}(c^2 - d^2) = \frac{1}{2}n^2 + dn = r^2n^2$:

whence $n = \frac{2d}{2r^2 - 1}$, $c = d + n = d + \frac{2d}{2r^2 - 1} = \frac{d(2r^2 + 1)}{2r^2 - 1}$,

and by means of these $1 + \frac{c^2 + d^2}{2} = 1 + \frac{d^2(4r^4 + 1)}{2r^2 - 1}$,

Put $d = s(2r^2 - 1)$; then $1 + \frac{d^2(4r^4 + 1)}{2r^2 - 1} = 1 + s^2(4r^4 + 1)$

= a square = $(ts + 1)^2 = t^2s^2 + 2ts + 1$: whence $s = \frac{2t}{4r^4 + 1 - t^2}$

where r and t must be so related as to give s a whole number.

Take $t = 2r^2$, then $s = 4r^2$, $d = s(2r^2 - 1) = 4r^2(2r^2 - 1)$, $n = 2d \div (2r^2 - 1) = 8r^2$, $c = d + n = 4r^2(2r^2 + 1)$, $a^2 = \frac{1}{2}(c^2 - d^2) = 64r^6$, and $(a+b)^2 = \frac{1}{2}(c^2 + d^2 + 2) = 64r^6 + 16r^4 + 1$, which are the two required squares, where r may be taken at pleasure.

Take $r = 1$, then the two squares are 81 and 64.

Take $r = 2$, and the two squares are 16641 and 4096.

This question was also answered by Dr. O'Riordan.

II. QUESTION 212, by INVESTIGATOR.

What sum should A, aged 30, pay on the event of his surviving B, aged 65, for an annuity of 1000*l.* during their joint lives, using the Northampton table of the probable duration of life, and allowing 5 per cent. interest of money?

SOLUTION, by INVESTIGATOR, the Proposer.

As this question contains in fact two distinct questions, viz. 1st. To find what the present value is of an annuity of £1000, to continue during the joint lives of A and B; and 2dly. What sum is equivalent to that present value, if deferred till the death of B, and limited also to the restriction of A's being then alive; so the complete solution may naturally be divided into two parts, each of which we shall consider in order.

1. Taking *a, b, c, d, e, &c.* to denote the number of the living at the age of A, and at the end of the 1st, 2d, 3d, 4th, &c. succeeding year; and *p, q, r, s, t, &c.* the number of the living corresponding to the age of B, to B's age + 1, + 2, + 3, + 4, &c. respectively, as found in Dr. Price's Northampton Table of the probable duration of human life, and putting *R* for the amount of £1 in a year, (i. e. 1.05 when it is 5 per cent. interest,) we shall have

The probability that

A will be alive at the end of the	B will be alive at the end of the	And that both A and B will be alive at the end of the
1st Year = $\frac{b}{a}$	$\frac{q}{p}$	$\frac{bq}{ap}$
2d = $\frac{c}{a}$	$\frac{r}{p}$	$\frac{cr}{ap}$
3d = $\frac{d}{a}$	$\frac{s}{p}$	$\frac{ds}{ap}$
4th = $\frac{e}{a}$	$\frac{t}{p}$	$\frac{et}{ap}$
&c&c.....	&c.....	&c.....

Therefore, by discounting the first term of the last rank of probabilities, namely, $\frac{bq}{ap}$, for one year, (i. e. multiplying it by R^{-1}); the second, $\frac{cr}{ap}$, for 2 years, (i. e. multiplying it by R^{-2}); the third,

third, $\frac{ds}{ap}$, for 3 years, (i. e. multiplying it by R^{-3}); the fourth $\frac{et}{ap}$, for 4 years, (i. e. multiplying it by R^{-4}), &c. and collecting the several products, and multiplying the whole sum by 1000, the present value of an annuity of £1000 upon the joint lives of A and B, is found to be equal to

£. $1000 \left(\frac{bq}{apR} + \frac{cr}{apR^2} + \frac{ds}{apR^3} + \frac{et}{apR^4} + \&c. \text{ to the end of the table, or till one of the factors in the numerator becomes } = 0, \right)$ or equal to $\frac{1000}{apR} \left(bq + \frac{cr}{R} + \frac{ds}{R^2} + \frac{et}{R^3} + \&c. \right)$ from which expression, the value is readily computed, and will be found to be = £6447.841.

2. Let the value of the annuity just found, (£6447.841,) be put = P, the equivalent contingent sum, that is to be paid in case A survives B, = Q, and let the same symbols as in the preceding investigation be retained, and we shall have the probability of

A's being alive at the end of the *first* year = $\frac{b}{a}$

A's dying in the first year = $\frac{a-b}{a}$

B's dying in the first year = $\frac{p-q}{p}$

A's surviving B in, and being alive at the end of the 1st year = $\frac{b}{a} \left(\frac{p-q}{p} \right)$

and the probability of their both dying in that year, and B dying first, nearly*, = $\frac{1}{2} \left(\frac{a-b}{a} \right) \left(\frac{p-q}{p} \right)$.

Again, the probability of

A's being alive at the end of the second year, is..... = $\frac{c}{a}$

A's dying in the second year = $\frac{b-c}{a}$

B's dying in the second year = $\frac{q-r}{p}$

A's surviving B in, & being alive at the end of the 2d year = $\frac{c}{a} \left(\frac{q-r}{p} \right)$

* I say nearly, for although our best writers on the subject have, as it were contemporaneously, taken this as the exact chance that is above mentioned, yet, of its differing from the truth in many cases, when the period of time is so long as a year, there cannot be entertained a moment's doubt.

and

and the probability of their both dying in that year, and B dying first, nearly, $= \frac{1}{2} \left(\frac{b-c}{a} \right) \left(\frac{q-r}{p} \right)$.

In like manner, the probability of

A's being alive at the end of the third year is $= \frac{d}{a}$

A's dying in the third year $= \frac{c-d}{a}$

B's dying in the third year $= \frac{r-s}{p}$

A's surviving B in, & being alive at the end of the 3d year $= \frac{d}{a} \left(\frac{r-s}{p} \right)$

and the probability of their both dying in the 3d. year, and B dying first, nearly, $= \frac{1}{2} \left(\frac{c-d}{a} \right) \left(\frac{r-s}{p} \right)$.

And also, the probability of

A's being alive at the end of the fourth year is $= \frac{e}{a}$

A's dying in the fourth year $= \frac{d-e}{a}$

B's dying in the fourth year $= \frac{s-t}{p}$

A's surviving B in, & being alive at the end of the 4th year $= \frac{e}{a} \left(\frac{s-t}{p} \right)$

and the probability of their both dying in the fourth year, and B dying first, nearly, $= \frac{1}{2} \left(\frac{d-e}{a} \right) \left(\frac{s-t}{p} \right)$.

&c.

&c.

Therefore, taking it as an equality of chance, that the event of A's surviving B, may happen in the *first* as in the *second half* of any one year, and allowing the proper discount on the two last mentioned probabilities in each of the above mentioned years, respectively, and collecting the several quantities, there will arise the following equation, viz. $P = Q \times$

$$\left\{ \begin{aligned} & \frac{b}{aR^{\frac{1}{2}}} \left(\frac{p-q}{p} \right) + \frac{c}{aR^{\frac{1}{2}}} \left(\frac{q-r}{p} \right) + \frac{d}{aR^{\frac{1}{2}}} \left(\frac{r-s}{p} \right) + \frac{e}{aR^{\frac{1}{2}}} \left(\frac{s-t}{p} \right) + \&c. \\ & + \frac{a-b}{2aR^{\frac{1}{2}}} \left(\frac{p-q}{p} \right) + \frac{b-c}{2aR^{\frac{1}{2}}} \left(\frac{q-r}{p} \right) + \frac{c-d}{2aR^{\frac{1}{2}}} \left(\frac{r-s}{p} \right) + \frac{d-e}{2aR^{\frac{1}{2}}} \left(\frac{s-t}{p} \right) + \&c. \end{aligned} \right\}$$

Or,

Or, by taking the sum of the two series, and dividing all by Q .

$$\frac{P}{Q} \text{ is } = \frac{(a+b)(p-q)}{2apR^{\frac{1}{2}}} + \frac{(b+c)(q-r)}{2apR^{\frac{1}{2}}} + \frac{(c+d)(r-s)}{2apR^{\frac{1}{2}}} + \frac{(d+e)(s-t)}{2apR^{\frac{1}{2}}} + \&c.$$

And, if we multiply the whole by $2apR^{\frac{1}{2}}$,

$$\frac{2PapR^{\frac{1}{2}}}{Q} = (a+b)(p-q) + \frac{(b+c)(q-r)}{R} + \frac{(c+d)(r-s)}{R^2} + \frac{(d+e)(s-t)}{R^3} + \&c.$$

and, consequently,

$$Q = 2PapR^{\frac{1}{2}} \div \left\{ (a+b)(p-q) + \frac{(b+c)(q-r)}{R} + \frac{(c+d)(r-s)}{R^2} + \frac{(d+e)(s-t)}{R^3} + \&c. \right\} = \mathcal{L}12224.69, \text{ or } \mathcal{L}12224.13s. 9\frac{1}{2}d. \text{ the sum that ought to be stipulated to be paid by A, at the death of B, for the annuity mentioned in the question.}$$

Remark 1. If we would avail ourselves of the assistance of the tables of annuities upon joint lives, the series that expresses the value of $\frac{P}{Q}$, when the multiplications are performed in the numerators of the several fractions of which it is composed, becomes equal to

$$\frac{1}{2apR^{\frac{1}{2}}} \times (ap + bp - aq - bq + \frac{bq + cq - br - cr}{R} + \frac{cr + dr - cs - ds}{R^2} + \frac{ds + es - dt - et}{R^3} + \&c.)$$

Which series is evidently equal to the four following, viz.

$$\frac{1}{2apR^{\frac{1}{2}}} \times \left\{ \begin{array}{l} (ap + \frac{bq}{R} + \frac{cr}{R^2} + \frac{ds}{R^3} + \&c.) \\ (bp + \frac{cq}{R} + \frac{dr}{R^2} + \frac{es}{R^3} + \&c.) \\ -(aq + \frac{br}{R} + \frac{cs}{R^2} + \frac{dt}{R^3} + \&c.) \\ -(bq + \frac{cr}{R} + \frac{ds}{R^2} + \frac{et}{R^3} + \&c.) \end{array} \right\}$$

Let α and ϕ denote the number of the living corresponding to 'A and 'B, two lives whose ages are less by one year than the ages of A and B, and let 'A'B, A'B, 'AB and AB denote the value of the joint lives of 'A and 'B, A and 'B, 'A and B, and A and B respectively, as taken from the tables of annuities upon joint lives;

then, by multiplying the first of the foregoing series by $\frac{ap}{\alpha\phi R^{\frac{1}{2}}}$,

the second by $\frac{p}{\phi R^{\frac{1}{2}}}$, the third by $\frac{a}{\alpha R^{\frac{1}{2}}}$, and the fourth by $\frac{t}{R^{\frac{1}{2}}}$, there

will

will be produced

$$\left. \begin{aligned} \frac{1}{2apR} \times \left(ap + \frac{bq}{R} + \frac{cr}{R^2} + \frac{ds}{R^3} + \&c. \right) \\ \frac{1}{2apR} \times \left(bq + \frac{cq}{R} + \frac{dr}{R^2} + \frac{es}{R^3} + \&c. \right) \\ - \frac{1}{2apR} \times \left(aq + \frac{br}{R} + \frac{cs}{R^2} + \frac{dt}{R^3} + \&c. \right) \\ - \frac{1}{2apR} \times \left(bq + \frac{cr}{R} + \frac{ds}{R^2} + \frac{et}{R^3} + \&c. \right) \end{aligned} \right\}$$

Of which expressions the first is evidently $= \frac{1}{2}A'B$, the second $= \frac{1}{2}A'B$, the third $= -\frac{1}{2}A'B$, and the fourth $= -\frac{1}{2}A'B$. Therefore, by dividing by the preceding factors, &c. Q is found

to be equal $P \div \frac{R^{\frac{1}{2}}}{2} (A'B \times \frac{ap}{ap} + A'B \times \frac{p}{p} - A'B \times \frac{a}{a} - AB)$; which is a very commodious theorem for obtaining

the value of Q by means of the tables of annuities upon joint lives and the original table of the probable duration of human life.

Remark 2. The value of Q , if computed by Mr. Simpson's Approximation, aided by Dr. Price's correction, (see Dr. P—'s Revers. Payments, vol. 1, pa. 37 and 38, 6th Edit.) comes out $= £12702$, exceeding the true value by $£477. 6s. 2d.$ If computed by the rule on pa. 40 and 41 of the same vol. it will be found to be $£12535$, exceeding the true value by $£310. 6s. 2d.$ If computed directly from the Northampton table, and discount be allowed on the several probabilities up to the end of each year respectively, (the common, though erroneous method of allowing discount,) it will be found $= £12525$, differing from the true value by $£300. 6s. 2d.$ A certain ingenious calculator makes it $£13059$, differing from the true value by above $£834!$

Rem. 3. Were we in possession of tables of the probable duration of human life for much shorter periods of time than years, and complete tables of annuities answering thereto, it would be easy to deduce a more exact value than even that which is exhibited by the above theorem, something similar to the ingenious method made use of in Art. xi, No. 7, of the New Series of the Repository, but considering the vague nature of the subject, and the tables now in common use, it is presumed, that the exactness of the above method of solution is abundantly sufficient for all practical purposes, and the ease with which the value of Q may be computed by means of the theorem in Rem. 1, cannot fail of recommending it in preference to some other *less correct and more troublesome* rules that have heretofore been given. This question was also answered by Mr. J. H. Harding, of the *Globe Insurance Office*.

III. QUESTION 213, by Mr. BAZLEY.

Find a circle whereof the diameter, and the chord, sine, and versed-sine of an arc of it, are all integers ?

SOLUTION, by Mr. CUNLIFFE.

Let sz denote the sine, and vz the versed sine of an arc ; then will $z\sqrt{(v^2 + s^2)}$ denote the chord, and $z(v^2 + s^2) \div v$ the diameter of the circle, which will all be rational, when $v^2 + s^2$ is a rational square. Put $v = 2mn$, and $s = m^2 - n^2$; then $z\sqrt{(v^2 + s^2)} = z(m^2 + n^2)$, and $z(v^2 + s^2) \div v = z(m^2 + n^2) \div 2mn$.

Put $z = 2mn$, then since $sz = 2mn(m^2 - n^2)$; versed sine $vz = 4m^3n^2$; chord $z\sqrt{(v^2 + s^2)} = 2mn(m^2 + n^2)$, and diameter $z(v^2 + s^2) \div v = (m^2 + n^2)^2$, where m and n denote any whole numbers.

The preceding expressions give only one arc that will answer to the same radius ; but any number of arcs of the same circle whose chords, sines, and versed sines will be rational, may be found as follows :

Put $m^2 + n^2 = m'^2 + n'^2 = m''^2 + n''^2$, &c. then by the known method of dividing the sum of two squares into two others, we may put

$$m' = \frac{m(p^2 - q^2) + 2npq}{p^2 + q^2} \text{ and } n' = \frac{n(p^2 - q^2) - 2mpq}{p^2 + q^2},$$

$$m'' = \frac{m(p'^2 - q'^2) + 2np'q'}{p'^2 + q'^2} \text{ and } n'' = \frac{n(p'^2 - q'^2) - 2mp'q'}{p'^2 + q'^2}, \text{ \&c.}$$

where supposing m and n given, $p, q; p', q', \text{ \&c.}$ may be taken at pleasure, but not in the same ratio. Then the diameter will be expressed by $(m^2 + n^2)^2$; the chords by $2mn(m^2 + n^2)$, $2m'n'(m^2 + n^2)$, $2m''n''(m^2 + n^2)$, &c. ; the sines by $2mn(m^2 - n^2)$, $2m'n'(m^2 - n^2)$, $2m''n''(m'^2 - n'^2)$, &c. and the versed sines by $4m^3n^2$, $4m'^3n'^2$, $4m''^3n''^2$ &c.

This question was also answered by Dr. O'Riordan.

IV. QUESTION 214, by AMICUS.

The ends A, C of two given beams are moveable about two fixed points, and the other ends B, D are connected by a string

of a given length which passes over a fixed pulley. It is required to determine the position of the beams when they are in equilibrium; the pulley and beams being in the same vertical plane?

SOLUTION, by Mr. J. JOHNSON, Birmingham.

Let P, (fig. 129, pl. 7,) be the point where the pulley is fixed, and draw AE, BG, CF, DH perpendicular to the horizontal line EPF. Also, PK, PL parallel to the beams AB, CD, meeting the lines BG, DH, produced, in K and L; and let Q, R, be the centres of gravity of the beams AB, CD; and W, w their respective weights. Then, by the property of the lever, the force which, acting in the direction BG, would sustain the end B, is to the whole weight of the beam, as AQ to AB, and is therefore equal to $(AQ \div AB) W$; which being resolved into a force acting in the direction of the string BP, is $= (BP \div BK) (AQ \div AB) W$. In like manner, the force acting in the direction of the string DP is $= (DP \div DL) (RC \div DC) w$; and when the beams are in equilibrium, these forces must be equal; that is,

$$\frac{BP}{BK} \times \frac{AQ}{AB} \times W = \frac{DP}{DL} \times \frac{RC}{DC} \times w.$$

Put $EP = a$, $AE = b$, $PF = c$, $CF = d$, $AB = m$, $AQ = n$, $DC = r$, $RC = s$, $GB = x$, $PG = y$, $PH = w$, $HD = z$, and l = the length of the string. Then $BP = \sqrt{(x^2 + y^2)}$, $GK = y(x - b) \div (a - y)$, and $BK = x + y(x - b) \div (a - y) = (ax - by) \div (a - y)$; therefore

$$\frac{BP}{BK} \times \frac{AQ}{AB} \times W \text{ is } = \frac{a - y}{ax - by} \times \sqrt{(x^2 + y^2)} \times \frac{n}{m} W.$$

And, in the same way we find

$$\frac{DP}{DL} \times \frac{RC}{DC} \times w = \frac{c - w}{cz - dw} \times \sqrt{(w^2 + z^2)} \times \frac{s}{r} w.$$

Therefore,

$$\frac{n}{m} W \cdot \frac{a - y}{ax - by} \cdot \sqrt{(x^2 + y^2)} = \frac{s}{r} w \cdot \frac{c - w}{cz - dw} \cdot \sqrt{(w^2 + z^2)}.$$

From this equation, and the equations $\sqrt{(x^2 + y^2)} + \sqrt{(w^2 + z^2)} = l$, $(a - y)^2 - (c - w)^2 = m^2$, and $(c - w)^2 + (z - d)^2 = r^2$ the value of the unknown quantities may be determined.

Solutions were also received from Messrs. Amicus, Cunliffe, and O'Riordan.

V. QUESTION 215, by HYPATIA.

This question has, by some mistake, been wrong proposed; it ought to have been as follows :

From A, one end of AB, the diameter of a circle, draw any chord AC. Bisect the arch AC in E, and join BE, meeting AC in F, and draw ED perpendicular to AB meeting AC in G. The lines AG, GF shall be equal. Required the demonstration?

SOLUTION, by Dr. O'RIORDAN.

Fig. 130, pl. 7. Because the arc AC is bisected in E, the angles CAE, EAB, AED are equal; therefore the right-angled triangles AEF, BDE, EDA are equiangular, or the angle GFE $\equiv$ to the angle GEF, and the angle GAE $\equiv$ to the angle GEA; therefore GF is $\equiv$ to GE, and AG $\equiv$ GE; consequently the lines AG, GF are equal.

Mr. J. Wallace, teacher of Mathematics, at Edinburgh, sent a neat solution to this question.

VI. QUESTION 216, by Mr. WALLACE, R. M. College.

Let a circle be given by position, and let A and B be two given points; a point C may be found, such, that if any straight line be drawn through it, meeting the circle in D and E, and AD, BD; also AE, BE be joined, there is a certain given space which is a mean proportional between $AD^2 + BD^2$ and $AE^2 + BE^2$. Required the demonstration?

SOLUTION, by Mr. LOWRY, R. M. College.

Fig. 131, pl. 7. Let the line joining the points A, B be bisected in P, and join PD, PE; then $AD^2 + BD^2 = 2AP^2 + 2PD^2$, and $AE^2 + BE^2 = 2AP^2 + 2PE^2$. Draw a line through P and the centre O, to meet the circle in H and K; and find the point V, so that $2OP \times OV$ may be equal to the square of the semi-diameter OH. Bisect PV in S, and draw DG, EN perpendicular to OH, then S is a given point; and by a well known theorem, $PD^2 = 2OP \times SG$, and $PE^2 = 2OP \times SN$.

Make $2OP \times SQ = 2AP^2$, then Q will also be a given point, and $AD^2 + BD^2 = 2AP^2 + 2PD^2 = 4OP \times QG$, and

$$AE^2 + BE^2 = 2AP^2 + 2PE^2 = 4OP \times QN.$$

2 D 2

Now

Now let $4OP \times l$ be the certain given space mentioned in the question, l being a line, the magnitude of which is yet to be determined;

then $4OP \times QG : 4OP \times l :: 4OP \times l : 4OP \times QN$,
or $QG : l :: l : QN$;

therefore $QG \times QN$ is $= l^2 =$ to a given space, and therefore, the point C must be so situated, that, in whatever direction DE is drawn, the rectangle of the distances QG, QN shall be equal to a given space. This we know, (from the lemma, on page 125 of this vol.) will be the case when the point C is in the line OQ , and when $QC^2 = QH \times QK$ is $=$ the given space l^2 , where the point C may be either within or without the circle. Hence this construction: Having found the point Q as in the analysis, take QC , on both sides of Q , a mean proportional between QH and QK ; then either of the points C will answer the conditions of the proposition, and $4OP \times PC$ will be the given space.

The demonstration is evident from the analysis, and the lemma referred to above.

The same algebraically.

Bisect AB , (fig. 132, pl. 7,) in F , and join DF, EF . Draw FG to the centre G , and bisect FG in I : Also, draw DH, EK, CL perpendicular to FG . Put $FG = a$, radius $DG = r$, $AF = d$, $CL = b$, $GL = c$, the angle $DOC = \phi$, $IH = x$, $IK = x'$, $GO = y$, and $p^2 =$ the given space.

Then $DF^2 - DG^2 = 2GF \times IH = 2ax$, or $DF^2 = 2ax + r^2$,
and $EF^2 - EG^2 = 2GF \times IK = 2ax'$, or $EF^2 = 2ax' + r^2$;

therefore $AD^2 + BD^2 = 2DF^2 + 2AF^2 = 4ax + 2r^2 + 2d^2$,

and $AE^2 + BE^2 = 2EF^2 + 2AF^2 = 4ax' + 2r^2 + 2d^2$;

or if $2r^2 + 2d^2$ be put $= 4ah$, we have $AD^2 + BD^2 = 4a(h+x)$

and $AE^2 + BE^2 = 4a(h+x')$; therefore by the question,

$$4a(h+x) : p^2 :: p^2 : 4a(h+x'),$$

$$\text{or, } (h+x)(h+x') = \left(\frac{p^2}{4a}\right)^2.$$

Now by trigonometry, $OD = y \cos \phi + \sqrt{(r^2 - y^2 \sin^2 \phi)}$,

and $OE = -y \cos \phi + \sqrt{(r^2 - y^2 \sin^2 \phi)}$;

therefore $OH = y \cos^2 \phi + \cos \phi \sqrt{(r^2 - y^2 \sin^2 \phi)}$,

and $OK = -y \cos^2 \phi + \cos \phi \sqrt{(r^2 - y^2 \sin^2 \phi)}$.

But IO is $= \frac{1}{2}a - y$, therefore

$h+x$ is $= h + \frac{1}{2}a - y \sin^2 \phi + \cos \phi \sqrt{(r^2 - y^2 \sin^2 \phi)}$, and

$h+x' = h + \frac{1}{2}a - y \sin^2 \phi + \cos \phi \sqrt{(r^2 - y^2 \sin^2 \phi)}$, therefore,

$$(h+x)(h+x') = (h + \frac{1}{2}a - y \sin^2 \phi)^2 - \cos^2 \phi (r^2 - y^2 \sin^2 \phi) = \left(\frac{p^2}{4a}\right)^2,$$

or

or, if m be put $= h + \frac{1}{2}a$, and $c + \frac{b \cos \varphi}{\sin \varphi} = GL + LO = y$, we have

$$m^2 - 2m \left(c + \frac{b \cos \varphi}{\sin \varphi} \right) (\sin^2 \varphi - r^2 + r^2 \sin^2 \varphi) + \left(c + \frac{b \cos \varphi}{\sin \varphi} \right)^2 \sin^2 \varphi = \left(\frac{p^2}{4a} \right)^2.$$

This equation contains all the conditions of the question; but since the angle φ is variable, we must determine what particular values must be assigned to the constant quantities, so that the equation may hold true whatever the value of φ may be. This

will evidently be the case when $b = 0$, and $m^2 - r^2 = \left(\frac{p^2}{4a} \right)^2$; for then the equation becomes $-2mc \sin^2 \varphi + r^2 \sin^2 \varphi + c^2 \sin^2 \varphi = 0$, in which φ may be any angle we please. Dividing the last equation by $\sin^2 \varphi$, and completing the square, we have $c = m \pm \sqrt{(m^2 - r^2)}$.

The result of this investigation, then is, first, that $b = 0$, or that the point C is in the line GF, and coincides with the point L.

Secondly, that $m^2 - r^2 = \left(\frac{p^2}{4a} \right)^2$, or $p^2 = 4a\sqrt{(m^2 - r^2)}$.

the given space. And, thirdly, that c or the distance of the point C from the centre G, is $=$ to $m - \sqrt{(m^2 - r^2)}$, or to $m + \sqrt{(m^2 - r^2)}$, which shews that there are two points C that answer the conditions of the question; the one being within and the other without the circle.

Solutions were likewise received from Messrs. Cunliffe and O'Riordan.

VII. QUESTION 217, by Mr. LOWRY.

An ellipse, and a straight line, being given by position; a point may be found such, that if any straight line be drawn through that point to meet the ellipse in two points, and perpendiculars be drawn from those points to the straight line given by position, the rectangle of those perpendiculars shall always be equal to a certain given space?

SOLUTION, by Mr. LOWRY, the Proposer.

LEMMA. If four points A, B, C, D, (fig. 133. pl. 7.) be taken in a straight line, so that the rectangle AB $\times$ AD may be equal to the square of AC, and let AE be taken equal to AC: Then

Then the line ED is harmonically divided in the points E, B, C, D.

And, conversely, if the line be harmonically divided in these points, and EC be bisected in A, the square of AC is equal to the rectangle AB $\times$ AD.

Because AB $\times$ AD is $= AC^2$, AB:AC = AE :: AC = AE:AD, and by composition and division,

$$EB : AE :: ED : AD,$$

$$BC : AE :: CD : AD;$$

therefore, EB : ED :: BC : CD, and the line ED is harmonically divided in the points E, B, C, D.

Conversely. Because EB : ED :: BC : CD, or EB : BC :: ED : CD, we have by division and composition,

$$2AB : BC :: 2AE = 2AC : CD,$$

$$2AE = 2AC : BC :: 2AD : CD;$$

therefore, AB : AC :: AC : AD, and AB $\times$ AD = AC².

This being premised. Let EDF, (fig. 134, pl. 7,) be the given ellipse, and AB the straight line given by position; and suppose that the point P is found, such, that if any straight line be drawn through it to meet the curve in E and F, and EG, FH be drawn perpendicular to AB, the rectangle EG $\times$ FH may be equal to a given space, which space is to be found.

In the first place let P be within the ellipse, and draw PQ perpendicular to AB: Then since the proposition is affirmed to be true for every position of EF, it must be true when EF is parallel to AB; and in this case, the rectangle of the perpendiculars is equal to the square of PQ; therefore, EG $\times$ FH must be equal to PQ², or PQ² must be the given space. Let FE be produced to meet AB in L; then by similar triangles,

$$EG : EL :: PQ : PL,$$

$$\text{and } FH : FL :: PQ : PL;$$

$$\text{therefore } EG \times FH : EL \times FL :: PQ^2 : PL^2;$$

but EG $\times$ FH is = to PQ², and therefore EL $\times$ FL is = PL².

On EL produced, take LM = PL; then by the lemma, the line MF is harmonically divided in the points M, E, P, F: wherefore if MN be drawn parallel to AB to meet the diameter DC, drawn through P in N, the position of the point P must be such, that any straight line drawn through it to meet the curve in E and F, and the line NM in M, shall be harmonically divided in the points M, E, P, F.

Now, we know, from the writers on Conic Sections, that this will be the case when NM is parallel to the ordinate of the diameter DC, and when the points P, N are so situated that the line NC is harmonically divided in the points N, D, P, C, that is,

by

by the lemma, (since ON is $= OP$), when $OD \times OC$ is $= OP^2$.

Secondly, when the point P is without the ellipse.

In this case the given space may be determined by considering the extreme case, when the line drawn through P is a tangent to the curve. Let lines be drawn from P , (fig. 135, pl. 7,) to touch the curve at K and O , and draw KO to meet PC in N , and PF in M . Also, let KV , MQ and OW be drawn perpendicular to AB . Then when the line PEF coincides with PK , the points E and F will coincide with K , and the rectangle of the perpendiculars will be equal to KV^2 . In like manner, when PEF coincides with PO , the rectangle of the perpendiculars will be $=$ to OW^2 ; but since the rectangle of the perpendiculars is $=$ to the same given space in all cases of the proposition, the square of KV must be equal to the square of OW , or $KV = OW$; therefore KO must be parallel to AB , and the rectangle $EG \times FH =$ to MQ^2 .

From similar triangles, as in the preceding case, we have $EL \times FL = LM^2$; and because OK is the line which joins the points of contact of the tangents drawn from P , the line PF is harmonically divided in the points P, E, M, F ; therefore, by the lemma, PM is bisected in L , and consequently PN is bisected in I : Also, PC is harmonically divided in the points P, D, N, C ; therefore, by the lemma, $ID \times IC$ is $= IN^2 = IP^2$. Hence we have this construction for both cases: draw the diameter DC so, that its ordinates may be parallel to AB , and let it meet AB in I . Take IP a mean proportional between ID and IC ; then P will be the point required, and PQ^2 the given space.

The demonstration is evident from the analysis.

The same analysis and construction will also apply to the hyperbola; but it is necessary to remark that the straight line given by position, must always meet the hyperbola; whereas in the ellipse, it must always be without, and in the parabola a tangent to the curve. In this case any point whatever in the diameter which passes through the point of contact, will answer the conditions of the proposition.

VIII. QUESTION 218, by Mr. LOWRY.

If the vertical angle and sum of the sides of a plane triangle remain constant, what is the equation of the curve to which the base is always a tangent?

SOLUTION

SOLUTION, by Mr. LOWRY, the Proposer.

Let the straight lines AG, AH, (fig. 136, pl. 7,) include the given angle, and let GDH be the curve required, such, that if a tangent be drawn at any point D, meeting the lines AG, AH in B and C; the sum of AB, AC may be equal to a given line a .

Draw DE, DF parallel to AH, AG; and put $x = AE = DF$, and $y = DE = AF$; then BE is $= -\frac{y\dot{x}}{\dot{y}}$ and FC $= -\frac{x\dot{y}}{\dot{x}}$, and we have $AB + AC = x - \frac{y\dot{x}}{\dot{y}} + y - \frac{x\dot{y}}{\dot{x}} = a$,

$$\text{or } x - \frac{y\dot{x}}{\dot{y}} + y - \frac{x\dot{y}}{\dot{x}} - a = 0,$$

the fluxionary equation of the curve.

This equation is of that singular kind that admits of two fluents or integrals, namely, a complete or general integral, containing a constant arbitrary quantity; and a particular integral into which no arbitrary quantity enters, and which is not comprised in the general integral*. This class of equations is generally resolved by first taking the fluxion, considering the fluxion of one of the variables as constant; thus, in the present case, by taking the fluxion of the above equation, considering $\dot{x}$ as constant, we have

$$\dot{x} - \frac{y''\dot{x} - \dot{y}\dot{x}\dot{y}}{\dot{y}^2} + \dot{y} - \frac{\dot{x}\dot{y} + \dot{y}\dot{x}}{\dot{x}} = 0,$$

$$\text{and by reduction, } \left(\frac{\dot{x}\dot{y}}{\dot{y}^2} + \frac{x}{\dot{x}}\right) \ddot{y} = 0;$$

therefore, either $\frac{\dot{x}\dot{y}}{\dot{y}^2} + \frac{x}{\dot{x}}$ must be $= 0$, or $\ddot{y} = 0$.

If $\frac{\dot{x}\dot{y}}{\dot{y}^2} + \frac{x}{\dot{x}}$ be $= 0$, then $\frac{\dot{x}}{\dot{y}} = -\sqrt{\left(\frac{x}{y}\right)}$; and this value of $\frac{\dot{x}}{\dot{y}}$ being substituted in the original fluxionary equation

* On the subject of particular integrals, see an ingenious work by Mr. Woodhouse, of Cambridge, entitled "The Principles of Analytical Calculation," page 97. And "Leçons sur le Calcul des Fonctions," par M. La Grange.

it becomes $x + y + y \sqrt{\frac{x}{y}} + x \sqrt{\frac{y}{x}} - a = 0$, from

whence we deduce $\sqrt{x} + \sqrt{y} = \sqrt{a}$, or $(a - x) + y = 2\sqrt{ay}$, an equation to the common parabola, and which is the curve required. As this equation does not contain any constant arbitrary quantity, it is necessarily the particular integral equation. We shall now shew how, from the equation $\ddot{y} = 0$, the complete integral may be derived; and here, because $\dot{x}$ is constant, we have by integration, $\dot{y} = A\dot{x}$, and $y = B + Ax$, where B is a constant arbitrary quantity, (namely, the value of y when $x = 0$), and A a quantity whose value is dependant on that of B . Now

when x is $= 0$, y is $= \frac{a}{1 - \frac{\dot{x}}{\dot{y}}} = \frac{a}{1 - \frac{1}{A}} =$ (by hy-

pothesis) to B ; or if B be put $= \frac{a}{b}$ (to render the expressions

homogeneous), we have $\frac{a}{1 - \frac{1}{A}} = \frac{a}{b}$, or $A = -\frac{1}{b-1}$

and the complete integral equation becomes $y = \frac{a}{b} - \frac{1}{b-1} x$,

or $\frac{b^2}{2} y - b y - b(a - x) + 2a = 0$, an equation to a straight line.

Indeed it is evident, that the straight line BC , will satisfy the conditions of the problem, provided it be placed so that $AB + AC$ may be equal to a ; that is, if b be taken of any value from 1 to ∞ , and AC be made $= \frac{a}{b}$, and $AB = a - \frac{a}{b}$; then $AB + AC$ will be equal to a , and the line BC will be the locus of the fluxionary equation $x - y \frac{\dot{x}}{\dot{y}} + y - x \frac{\dot{y}}{\dot{x}} - a = 0$, as well as the parabola GDH . In fact the pa-

rabola is formed by the continual intersections of the straight lines represented by the complete integral equation, when the arbitrary quantity b is made continually to vary.

Though the particular integral cannot be derived from the general one, by assigning any particular value to the arbitrary

quantity b ; yet it may be deduced from it by considering the quantity b as variable, or a function of x and y .

Thus, if in the complete integral equation $\frac{b^2}{2} y + by - b(a-x) + 2a = 0$, we take the fluxion, supposing b only to vary, we have $b = 1 + \frac{a-x}{y}$; this value of b being substituted in the given equation, it becomes, after proper reduction,

$$(a-x)^2 + 2y(a-x) + y^2 = 4ay,$$

or $(a-x) + y = 2\sqrt{ay}$, the particular integral as determined before.

This question was also answered by Dr. O'Riordan.

IX. QUESTION 219, by Mr. WALLACE.

Let AB, AC be two straight lines given by position, and P a given point: it is required to draw a line through P to meet the given lines in D and E, so that $AD \times AE$ may be to $DP \times PE$ in a given ratio.

FIRST SOLUTION, by Dr. O'Riordan.

Suppose that DE, (fig. 137, pl. 7.) is drawn as required, and let PI, PH be drawn perpendicular to AB, BC: and AQ perpendicular to DE; then PI, PH are given lines, and since

$AD \times IP = DP \times AQ =$ twice the triangle DPA,
and $AE \times PH = PE \times AQ =$ twice the triangle PAE,
we have $AD : DP :: AQ : IP$, and $AE : PE :: AQ : PH$;

therefore $AD \times AE : DP \times PE :: AQ^2 : IP \times PH$, or AQ^2 has to $IP \times PH$ the given ratio of $AD \times AE$ to $DP \times PE$: Let this ratio be taken = that of PS to PI, and then AQ^2 is = to the given space $PS \times PH$. Hence this construction: On AP describe a semicircle, and therein apply the line $AQ =$ a mean proportional between PS and PH: and through the points P and Q draw DE.

SECOND SOLUTION, by Mr. CUNLIFFE.

Join PA, (fig. 138, pl. 7.) and draw PQ parallel to AB meeting CA in Q. Put $AQ = a$, $QP = b$, $AE = x$, and PE

$= y$: also let the given ratio of $AD \times AE$ to $PD \times PE$ be denoted by that of b to c . Then because of the parallels QP, AB ,

$$QE : QP :: AE : AD = QP \times AE \div QE;$$

$$\text{whence } AD \times AE = QP \times AE^2 \div QE;$$

$$\text{Also, } QE : PE :: QA : PD = QA \times PE \div QE;$$

$$\text{whence } PD \times PE = QA \times PE^2 \div QE; \text{ and therefore}$$

$$AD \times AE : PD \times PE :: QP \times AE^2 : QA \times PE^2 :: b : c: \text{ Whence}$$

$$c \times QP \times AE^2 = b \times QA \times PE^2, \text{ that is, } bcx^2 = aby^2, \text{ or } cx^2 = ay^2.$$

Hence it appears, that x to y , or AE to PE have a given ratio, which makes the construction very easy, and may be as follows:

Take two right lines Ae and ep in the given ratio of AE to EP , and set off Ae upon AC ; from e to AP apply ep , and through P draw PDE parallel to pe cutting AB, AC in D and E , and the thing is done.

X. QUESTION 220, by Mr. LOWRY.

If through a given point P on the surface of a sphere, two great circles be drawn to intersect a small circle given by position in the points A, B and C, D ; and the points AC, BD be joined by great circles which intersect in Q . What is the locus of the point Q ?

SOLUTION, by Mr. LOWRY, the Proposer.

Let O , (fig. 139, pl. 7,) be the centre of the sphere, and $ABCD$ the small circle given by position. Conceive lines to be drawn from the points A, B, C, D to the centre of the sphere, and let the planes of the great circles PAB, PDC, AQC, BQD meet the plane of the small circle in the straight lines GAB, GCD, AIC and DIB respectively. Then because P is a given point, G is also a given point; and, since straight lines are drawn from G to meet the circle in the points A, B and A, D , the locus of the intersection I of the lines AC, BD is a straight line given by position; namely, the line EF which joins the points of contact of tangents drawn to the circle from G . This property of the circle is too generally known to need demonstrating here. Let a plane pass through the line EF and the centre O ; then since the point I is always in the line EF , the planes AIC, BID will always intersect in the plane EFO ; therefore the point Q must be in the curve formed by the intersection of the plane OEF with the superficies of the sphere; that is, in the great circle which passes through the points E, F . And since the planes GEO, GFO

$\simeq E \simeq$

touch

touch the circle at E, and F, the arches PE, PF, which are in these planes, will also touch the small circle in these points; therefore if great circles be drawn from the given point to touch the small circle in the points E and F, a great circle described through these points is the locus required.

It is obvious that the great circles joining the points A, D; B, C will also intersect in the great circle EQF.

In the figure referred to, the given point is without the circle, but the same reasoning will apply when it is within the circle; only the locus will then be on the other side of S. Thus, if Q be the given point, the locus of P will be a great circle drawn perpendicular to the diameter which passes through Q.

Conceive a plane passing through the points G, O, and any point I, in the line EF, to meet the plane of the circle in the straight line GSIT, and the surface of the sphere in the great circle PSQT; then it is well known, that the line GT is harmonically divided in the points G, S, I, T, or $GS : GT :: SI : IT$; we shall now shew that the arch PT is divided, so that $\sin PS : \sin PT :: \sin SQ : \sin QT$.

This division may, for the sake of abridging language, be called the harmonical division of the arch PT.

Let lines be drawn from the points S, Q, T to the centre of the sphere: Then by trigonometry,

$$\begin{aligned} GS : SO &:: \sin GOS : \sin OGS, \\ OT = SO : GT &:: \sin OGS : \sin TOG; \text{ therefore ex æquo,} \\ GS : GT &:: \sin GOS : \sin TOG :: \sin PS : \sin PT. \end{aligned}$$

In the same way, it is shewn, that

$$SI : IT :: \sin SQ : \sin QT.$$

Therefore $\sin PS : \sin PT :: \sin SQ : \sin QT$.

This harmonical division of the arch PT suggests a number of curious properties analogous to those which are connected with the harmonical division of the diameter of a circle in *plano*. The following are some of the most remarkable.

On the great circle AB, (fig. 140, 141, pl. 7,) which passes through the pole C of a small circle on the surface of the sphere, let two points D, P, be taken, so, that $\sin PA : \sin PB :: \sin AD : \sin DB$, or which is the same thing, (as is easily proved,) that $\tan PC \times \tan CD$ may be $= \tan^2 CA$, and through D let a great circle EF be drawn perpendicular to AB: Then,

1st. If any great circle be drawn through P to meet the small circle in S and T, and the great circle EF in Q, the arc PT is harmonically divided in the points P, S, Q, T.

2nd. If great circles be drawn through the points S, D; T, D to meet the small circle in G and K; the angle PDS is $\equiv$ BDT; and the angle SPD $\equiv$ DPG.

3rd. The points P, G, K are in the arch of a great circle.

4th. The arch AS is $\equiv$ to the arch AG, and the arch ST $\equiv$ to the arch GK.

5th. $\sin PS : \sin PT :: \sin SD : \sin DT$.

and $\sin PS : \sin PK :: \sin SD : \sin DK$.

6th. If great circles be drawn to touch the small circle at S and T, they will intersect in the great circle EF.

Dr. O'Riordan also answered this question.

XI. QUESTION 221, by Mr. LOWRY.

Find the perspective representation of a given spheroid placed on a horizontal plane; the relative positions of the eye, the vertical plane of the picture, and the spheroid being given?

SOLUTION, by Mr. LOWRY, the Proposer.

Let E, (fig. 142, pl. 7.) be the position of the eye, and C the centre of the given spheroid. Conceive a straight line to be drawn through the points E, C to meet the surface of the spheroid in A and B; and let a point O be found in AC, such, that the rectangle $OC \times CE$ may be equal to AC^2 . Through O, let a plane be drawn parallel to the tangent plane at A, meeting the surface of the spheroid in the curve IHGF, which, it is well known, is an ellipse, except in the particular case when the plane is perpendicular to AB, and then it is a circle. Let AGB be any section of the spheroid, made by a plane passing through the eye, and the centre, and meeting the ellipse (or circle) IHGF in G; and join EG, GO. Then because the plane HGOF is parallel to the tangent plane at A, it is evident that GO will be parallel to the tangent of the ellipse AGB at A; and since the rectangle $OC \times CE$ is $\equiv$ to AC^2 , the line EG will touch the ellipse AGB at G, (Hamilton's Conic Sections, b. 1, p. 48). Whence it is evident that the ellipse IHGF is the locus of the points of contact of all the tangents drawn to the spheroid from E; or it is the curve of contact formed on the surface of the spheroid by the visual rays from E; and consequently its representation on the given plane will form the contour of the perspective of the spheroid. Now it is shewn by the writers on perspective, that the representation of any ellipse, on a given plane,

is a conic section, and therefore the perspective representation of the given spheroid will also be a conic section; namely, an ellipse when the spheroid is without the directing plane; a parabola when it touches the directing plane on the side next to the picture; and a hyperbola when it is cut by the directing plane.

Though it may be difficult to describe the ellipse HGF, on the surface of the spheroid, so that the common rules for putting an ellipse into perspective, may be applicable; yet we may easily determine as many points in the curve as shall be necessary for describing its representation on the given plane.

Let X represent the greater, and Y the less axis of the spheroid when drawn through the centre, parallel to the horizon; and conceive a vertical plane passing through E and C, to cut the spheroid in the elliptic section AFBH: Then because the point E, and the spheroid, are given by position, the vertical plane AFBH will make a given angle with the vertical plane, passing through the axis X, and therefore the ellipse AFBH will be given in magnitude and position. And, if tangents EF, EH be drawn from E, we may easily determine the points of contact F, H by first describing the ellipse on a plane, and drawing the tangents, and then transferring the points of contact to the surface of the spheroid. In like manner, if a plane pass through E, and each of the axes X, Y, cutting the spheroid in two elliptic sections, the points of contact of tangents drawn to these sections from E, may be found as before, and we shall then have six given points in the ellipse IHGF to describe its perspective; and the representation of any five of them being made on the plane of the picture, we have only to describe a conic section through these five points and it will be the figure required,

A solution was also received from Dr. O'Riordan.

XII. QUESTION 222, by Mr. BAZLEY.

It is required to determine the locus of the vertex of a plane triangle, whereof the radius of the inscribed circle, and difference of the angles at the base are given; the middle of the base being a fixed point in a straight line given by position?

FIRST SOLUTION, by Mr. BAZLEY, the Proposer.

CONSTRUCTION. Draw the indefinite right line MN, (fig. 143, pl. 7,) wherein take E for the middle of the base. Erect the perpendicular EF to the diameter of the inscribed circle and make the $\angle FEQ = \frac{1}{2}$ the difference of the angles at the base:
through

through F draw FVS parallel to MN cutting EQ in V; take VG (in VQ) $= \frac{1}{2}EV$, and GH, parallel to MN, $= \frac{1}{2}FV$. Lastly, to the asymptotes VQ, VS, through the point H, describe the hyperbola HR, which is the locus required.

DEMONSTRATION. Take any point C in HR, and draw CTOI parallel to EVQ, cutting VS in T, and MN in I; bisect TI in O, and drop the perpendicular OL on MN: with radius OL and centre O, describe the circle OL, and from C draw CA, CB touching it, and meeting MN in A and B. Then it is plain, that the triangles IOL, VEF are similar, and since $IO = \frac{1}{2}IT = \frac{1}{2}EV$, therefore $OL = \frac{1}{2}EF$: whence it is evident, that in the triangle ABC, OL is the given inscribed circle; and it is well known that IOL ($=FEV$) $= \frac{1}{2}$ the difference of the angles at the base. Now it only remains to prove that E is the middle of AB.

By the property of the hyperbola $VT \times TC = VG \times GH$, that is, by construction, $EI \times TC = IO \times IL$, or $EI:IL::IO:TC$, and by composition, $EI:EL::(IO:OC) IL:LK$; and, again, by composition, $EI:EL::EL:EK$; therefore by a known property, (whose demonstration may be seen at pa. 285, vol. 1, Old Series of the Repository,) E is the middle of the base AB.

SECOND SOLUTION, by Dr. O'RIORDAN.

Let ACB, (fig. 144, pl. 7.) be a triangle circumscribed by a circle, EF being the diameter which bisects the base AB in M: Draw EC, EB: upon EC take $Es = EB$, and draw st perpendicular to AB. Then the following things are pretty generally known; namely, s is the centre, and st the radius of the circle inscribed in the triangle ACB; also $\frac{1}{2}(\angle ABC - \angle BAC) = \angle FEC = \angle Lst$ which is given by the question. Therefore put $Lt = a$, $Ls = b$, $ML = x$, and $LC = v$. Then per similar triangles Lts , LME ,

$$Lt : Ls :: ML : EL = (Ls \times ML) \div Lt = bx \div a;$$

$$\text{whence } Es = EB = EL + Ls = \frac{bx}{a} + b, EC = EL + LC = \frac{bx}{a} + v.$$

Again, per similar triangles, ELB , EBC , $EL:EB::EB:EC$;

$$\text{whence } EL \times EC = EB^2 = Es^2, \text{ that is } \frac{bx}{a} \times \frac{bx+av}{a} = \frac{b^2}{a^2} \times (x+a)^2$$

or

or $x(bx + av) = b(x + a)^2$, which being reduced, gives $vx - 2bx = ab$.

Put $y = v - 2b$, and the equation becomes $xy = ab$, which expresses the known property of the co-ordinates of a hyperbola, with respect to its asymptotes; therefore the locus of the vertex C of the triangle is an hyperbola: The position of the asymptotes, &c. may be determined as follows:

Draw MP parallel to LC, on LC take $sl = sL$, and through l draw RQ parallel to AB, meeting MP in R. Then $RL = ML = x$, and $lC = LC - lL = v - 2b = y$; therefore $xy = ab$, or $Rl \times lC = Ls \times Ls$, which is the known property of an hyperbola, whose asymptotes are RP, RQ, and centre R. Suppose $x = y$, then $xy = y^2 = ab$, or $x = y = \sqrt{ab}$, which determines the place of the principal vertex.

XIII. QUESTION 223, *by* ———.

Let a circle be given by position, and let A and B be two given points, a point C may be found, such, that if any straight line be drawn through C to meet the circle in D and E, and AD, BD, AE, BE be joined, $AD^2 + DB^2 : AE^2 + EB^2 :: DC : CE$. Required the demonstration?

FIRST SOLUTION, *by* Mr. I. POUND.

ANALYSIS. Join AB, (fig. 145, pl. 7,) which bisect in G, and join DG, EG; then $AD^2 + DB^2 = 2AG^2 + 2DG^2$ and $AE^2 + EB^2 = 2GE^2 + 2GB^2$; and if GC', (C' being the centre of the circle,) be joined, meeting the circle in H and K, and $GL^2 = HLK$, and DM, EN be perpendicular to GC': Then by the 4th Porism of Euclid's 3rd. Book of Porisms, $DG^2 = 2GC' \times LM$, and $GE^2 = 2GC' \times LN$. Now make $4AG^2 = 2GC' \times LO$, then $2AG^2 + 2DG^2 = 4GC' \times OM$ and $2GE^2 + 2GB^2 = 4GC' \times ON$; therefore $OM : ON :: DC : CE$, that is, drawing RO perpendicular, and DQ, EP, parallel to CG', $DQ : PE :: DC : CE$; but if DE meet OR in R, $DQ : PE :: DR : RE$; therefore, finally, $DR : RE :: DC : CE$.

Whence it is evident, that C is such a point in OC' that $HO : OK :: HC : CK$. Q. E. I.

The construction and demonstration are evident from the analysis.

N. B.—It is evident that the porism is true for any figure given in species as well as squares.

SECOND

SECOND SOLUTION, by Mr. LOWRY.

By making the same construction and substitution, as in the solution to quest. 216, (fig. 132, pl. 7,) we have $AD^2 + BD^2 = 4a(h+x)$, and $AE^2 + BE^2 = 4a(h+x')$; therefore, by the question,

$$4a(h+x) : 4a(h+x') :: CD : CE,$$

$$\text{or } h+x : h+x' :: CD : CE;$$

and by composition and division,

$$(h+x) + (h+x') : (h+x) - (h+x') :: CD+CE : CD-CE.$$

But it is evident, from the solution alluded to, that

$$(h+x) + (h+x') = 2h+a-2y\sin^2\phi,$$

$$(h+x) - (h+x') = 2\cos\phi\sqrt{(r^2-y^2\sin^2\phi)}$$

$$CD+CE = 2\sqrt{(r^2-y^2\sin^2\phi)}$$

$$\text{and } CD-CE = -\frac{2b}{\cos\phi} + 2y\cos\phi;$$

therefore,

$$(h+\frac{a}{2}-y\sin^2\phi)(y\cos\phi-\frac{b}{\cos\phi}) = \cos\phi(r^2-y^2\sin^2\phi);$$

$$\text{or } (h+\frac{a}{2})y\cos^3\phi - (h+\frac{a}{2}-y\sin^2\phi)b = r^2\cos^3\phi;$$

in which equation, the angle ϕ must be indeterminate; which it evidently will be when b is $= 0$; for then y is $= c$, and the equation becomes $(h+\frac{1}{2}a)c\cos^3\phi = r^2\cos^3\phi$, or $(h+\frac{1}{2}a)c = r^2$. Therefore the point C is in the line GF, and at a

distance from G $=$ to $\frac{r^2}{h+\frac{1}{2}a}$.

THIRD SOLUTION, by Mr. CUNLIFFE.

Draw the line AB, (fig. 146, pl. 7,) which bisect in M, and draw MD, ME; also, draw MO, to the centre O of the given circle. By a known property

$AD^2+BD^2=2(AM^2+MD^2)$ and $AE^2+BE^2=2(AM^2+ME^2)$; therefore,

$$AD^2+BD^2 : AE^2+BE^2 :: AM^2+MD^2 : AM^2+ME^2 :: DC : CE.$$

Now a little attention to the more obvious circumstances, of the question will be sufficient to shew, that if there is such a

point C as that asserted, it must be situated somewhere in the line MO; for when $DC = CE$, then it is manifest from the preceding analogy, that $MD = ME$, and therefore the angles OCD , MCD are right-angles Euc. 3. III, and consequently the points O, C, M, are in the same right line.

This consideration will tend very much to simplify the investigation. Draw the radii OD, OE; also draw Dn , Em perpendicular to MO.

Put $OM = a$, $OC = b$, $AM = BM = c$, $OD = OC = r$, $ON = x$, and $OM = y$. Then, $Mn = a - x$, $Mm = a + y$, $Dn^2 = r^2 - x^2$, $Em^2 = r^2 - y^2$, $MD^2 = Mn^2 + Dn^2 = (a-x)^2 + r^2 - x^2 = a^2 + r^2 - 2ax$, $ME^2 = Mm^2 + Em^2 = (a+y)^2 + r^2 - y^2 = a^2 + r^2 + 2ay$: hence $AM^2 + MD^2 = a^2 + r^2 + c^2 - 2ax = s^2 - 2ax$, and $AM^2 + ME^2 = a^2 + r^2 + c^2 + 2ay = s^2 + 2ay$, putting $s^2 = a^2 + r^2 + c^2$. Now, per similar triangles, CnD , CmE , $DC : CE :: Cn : Cm$; therefore, $AM^2 + MD^2 : AM^2 + ME^2 :: Cn : Cm$, whence $Cm \times (AM^2 + MD^2) = Cn \times (AM^2 + ME^2)$, that is $(b+y) \times (s^2 - 2ax) = (x-b) \times (s^2 + 2ay)$, which gives

$$x = \frac{2bs^2 + (s^2 + 2ab)y}{s^2 + 2ab + 4ay}.$$

Again, per similar triangles, CnD , CmE , $Dn^2 : Em^2 :: Cn^2 : Cm^2$; whence

$$Dn^2 \times Cm^2 = Em^2 \times Cn^2, \text{ or } (r^2 - x^2) \times (b+y)^2 = (r^2 - y^2) \times (x-b)^2$$

$$\text{which gives } x = \frac{2br^2 + (r^2 + b^2)y}{r^2 + b^2 + 2by}.$$

$$\text{Therefore, } \frac{2bs^2 + (s^2 + 2ab)y}{s^2 + 2ab + 4ay} = \frac{2br^2 + (r^2 + b^2)y}{r^2 + b^2 + 2by},$$

which cleared of fractions, gives $2bs^2(r^2 + b^2) + (r^2 + b^2) \times (s^2 + 2ay)y + 4b^2s^2y + 2by^2(s^2 + 2ab) = 2br^2(s^2 + 2ab) + (r^2 + b^2)(s^2 + 2ab)y + 8abr^2y + 4ay^2(r^2 + b^2)$,

which equation will, in general, give a determinate value of y , but we want such a relation of the other quantities as will make y indeterminate; and this will be done by equating the like terms on each side. Beginning with the co-efficients of y^2 , $b(s^2 + 2ab) = 2a(r^2 + b^2)$; whence $b = 2ar^2 \div s^2$. And the coefficients of y being equated, give exactly the same result; as do also the remaining terms.

XIV. QUESTION 224, by Mr. IVORY.

Supposing x to denote any fraction less than unit, it is required to assign the value of the infinite product $(1+x)(1+x^2)(1+x^4) \&c.$

FIRST

FIRST SOLUTION, by Mr. J. WALLACE, teacher of Mathematics, Edinburgh.

Let P be the product required. Then, by the logarithmic series, we have

$$\begin{aligned}\text{Log. } (1+x) &= M(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8 \&c.) \\ \text{Log. } (1+x^2) &= M(\quad + x^2 \quad - \frac{1}{2}x^4 \quad + \frac{1}{3}x^6 \quad - \frac{1}{4}x^8 \&c.) \\ \text{Log. } (1+x^4) &= M(\quad \quad + x^4 \quad \quad - \frac{1}{2}x^8 \&c.) \\ \text{Log. } (1+x^8) &= M(\quad \quad \quad + x^8 \&c.) \\ &\&c. \quad \quad \quad \&c. \quad \quad \quad \&c.\end{aligned}$$

Hence, taking the sum, we have

$$\text{Log. } P = M(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8 + \&c.)$$

But $M(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \&c.)$ is known to be the logarithm of $\frac{1}{1-x}$. Hence it follows that $P = \frac{1}{1-x}$.

SECOND SOLUTION, by Mr. WALLACE, R. M. College.

$$\begin{aligned}\text{Because } (1+x)(1-x) &= 1-x^2 \\ (1+x^2)(1-x^2) &= 1-x^4 \\ (1+x^4)(1-x^4) &= 1-x^8 \\ &\dots\dots\dots \\ (1+x^n)(1-x^n) &= 1-x^{2n},\end{aligned}$$

therefore taking the products of the two sides of these identical equations, and leaving out the factors common to both, we get $(1+x)(1+x^2)(1+x^4)\dots\dots\times(1+x^n)(1-x)=1-x^{2n}$, and hence,

$$(1+x)(1+x^2)(1+x^4)\dots\dots\times(1+x^n) = \frac{1-x^{2n}}{1-x}.$$

This is true, whatever be the number of factors in the product. If, however, we suppose that number to be infinite, so that n is infinitely great, then $x^{2n} = 0$, (x being supposed less than unity), and consequently,

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\&c. = \frac{1}{1-x}.$$

THIRD SOLUTION, by ———.

$$\text{Let } f(x) = (1+x)(1+x^2)(1+x^4)\&c.$$

$$\text{Then } f(x^2) = (1+x^2)(1+x^4)(1+x^8)\&c.$$

2 F 2

Therefore

Therefore $f(x) = (1+x) f(x^2)$,

and $(1-x) f(x) = (1-x^2) f(x^2)$.

In like manner,

$$(1-x^2) f(x^2) = (1-x^4) f(x^4)$$

$$(1-x^4) f(x^4) = (1-x^8) f(x^8)$$

$$(1-x^8) f(x^8) = (1-x^{16}) f(x^{16})$$

&c.

&c.

Therefore, ultimately, $(1-x) f(x) = 1$,

$$\text{and } f(x) = \frac{1}{1-x}.$$

FOURTH SOLUTION, communicated by PHILALETES CANTABRIGIENSIS.

About twenty years have elapsed since the person alluded to in page 56. of vol. 1, of the *New Series* of this *Repository*, produced the following equations, viz.

Let x be less than 1, then by multiplying both numerator and denominator by equal quantities, we shall have

$$\frac{1}{1-x} = \frac{1+x}{1-xx} = \frac{(1+x)(1+xx)}{1-xx^4} = \frac{(1+x)(1+xx)(1+xx^4)}{1-xx^8}, \text{ \&c.}$$

Hence, putting n to denote any power of 2, which exceeds 8, we have

$$\frac{1}{1-x} = \frac{(1+x)(1+xx)(1+x^4)(1+x^8) \dots (1+xx^{2^{n-1}})}{1-xx^{2^n}};$$

and this equation, when n is infinite, becomes

$$\frac{1}{1-x} = (1+x)(1+xx)(1+x^4)(1+x^8)(1+x^{16}), \text{ \&c. ad infi.}$$

nitum, the term xx^{2^n} , in that case, becoming less than any assignable quantity, and consequently the denominator may be considered as $= 1$.

Corol. 1. It is evident that any power, or root, of the fraction on the first side of the equation now obtained, is equal to the continued product of the same power, or root, of all the factors on the second side of it.

Corol.

Corol. 2. It is evident also, that the

Logarithm of $\frac{1}{1-x} = L.(1+x) + L.(1+xx) + L.(1+x^3),$

&c. *ad infinitum.*

Scholium. Several other useful deductions may be made from the above equation; which equation, I observe, may easily be deduced from what was shown by the Rev. *John Hellins*, in the *Philosophical Transactions* of the Royal Society of London, for the year 1798, page 185.

XV QUESTION 225, by Mr. CUNLIFFE.

To find integer values for the sides of a trapezium that may be inscribed in a circle, such, that its diagonals, diameter of the circumscribing circle, and area, may all be expressed by whole numbers?

SOLUTION, by Mr. CUNLIFFE, the Proposer.

Let ABCE, (fig. 147, pl. 7.) be a trapezium inscribed in a circle, AC and BE the diagonals intersecting in M, and BF perpendicular to AC. Put $AB = m^2 + n^2$, $AF = m^2 - n^2$; $BM = m^2 + n^2$, $MF = m^2 - n^2$; $BC = m'^2 + n'^2$, $CF = m'^2 - n'^2$; whence $BF = 2mn = 2m'n' = 2m'n''$.

Take $BF = 24$, then $mn = m'n' = m''n'' = 12 \times 1 = 4 \times 3 = 6 \times 2$; therefore, we may take $m = 12$, $n = 1$; $m' = 4$, $n' = 3$; $m'' = 6$, $n'' = 2$; then $AB = m^2 + n^2 = 145$, $AF = m^2 - n^2 = 143$, $BM = m^2 + n^2 = 145$, $MF = m^2 - n^2 = 143$; $BC = m'^2 + n'^2 = 40$, $CF = m'^2 - n'^2 = 32$; and hence $AC = 175$, $AM = 136$ and $CM = 39$.

And by the property of the circle, $EM = \frac{AM \times MC}{BM} = \frac{136 \times 39}{145}$, hence $BE = BM + ME = \frac{5929}{25}$; and per similar triangles, BMA, CME; $BM : BA :: CM : CE = \frac{BA \times CM}{BM} = \frac{145 \times 39}{145} = \frac{1131}{5}$. Also, per similar triangles, CMB, EMA; $BM : BC :: AM : AE = \frac{BC \times AM}{BM} = \frac{40 \times 136}{25} =$

$$= \frac{1088}{5}. \text{ Again, by a known property, } \frac{AB \times BC}{BF} = \frac{145 \times 40}{24} \\ = \frac{725}{3} = \text{the diameter of the circumscribing circle.}$$

And by reducing the preceding numbers to a common denominator, and omitting it, we shall have $BF = 24 \times 75 = 1800$, $AB = 145 \times 75 = 10875$, $BC = 40 \times 75 = 3000$, $AC = 175 \times 75 = 13125$, $BE = 17787$, $CE = 1131 \times 15 = 16965$, $AE = 1088 \times 15 = 16320$, and the diameter of the circumscribing circle $= 725 \times 25 = 18125$.

The area of the trapezium may be obtained as follows: $AC \times \frac{1}{2}BF = 13125 \times 12 = 157500 = \text{area of the } \triangle ABC$. And it is pretty evident that $BM : BE :: \triangle ABC : \text{trapez.}$

$$ABCE = \frac{BE \times ABC}{BM} = \frac{17787 \times 13125 \times 12}{25 \times 75} = 17787 \\ \times 7 \times 12 = 1494108.$$

XVI. QUESTION 226, by Mr. CUNLIFFE.

Suppose a heavy hollow cylinder, of an uniform thickness, to roll freely along an inclined plane of a given length and inclination: having given the external diameter of the cylinder together with the time of rolling along the inclined plane, it is required from thence to determine the internal diameter of the cylinder?

SOLUTION, by Mr. CUNLIFFE, the Proposer.

Let SAB , (fig. 148, pl. 7.) be a section of the cylinder perpendicular to its axis; C its centre; CS and CR the external and internal radii of the section. Before we proceed to the solution of the question, it will be necessary to determine the centre of oscillation of the section, the point of suspension being in the circumference of the outer circle, and the section supposed to vibrate in a vertical plane edgeways.

Let Cq be an intermediate circle between CS and CR , and suppose O , in the diameter SB , the centre of oscillation of the section with respect to the centre of suspension S .

Put $CS = R$, $CR = r$, $Cq = x$, $p = 3.1416$, and let s denote the area of the annulus included between the circles, whose radii are Cq and CR . Then, by Emerson's Fluxions, page

$$311, CO = \frac{\int x^2 s}{R s} \text{ when } x = R. \text{ Now, } x^2 s' = x^2 \times 2px =$$

$$2px^3,$$

$2px^2\dot{x}$, the correct fluent of which is $\frac{1}{2}p \times (x^4 - r^4)$; also $s = p(x^2 - r^2)$; therefore $\frac{\int x^2 \dot{s}}{Rs} = \frac{\frac{1}{2}p(x^4 - r^4)}{Rp(x^2 - r^2)} = \frac{x^2 + r^2}{2R}$, which, when $x = R$ becomes $\frac{R^2 + r^2}{2R}$, that is, $CO = \frac{R^2 + r^2}{2R}$, and $SO = SC + CO = R + \frac{R^2 + r^2}{2R} = \frac{3R^2 + r^2}{2R}$.

Solution of the Question.

Let l denote the length, and h the height of the inclined plane, and put $g = 16\frac{1}{2}$; also let t denote the given time of the cylinder's rolling along the inclined plane.

Then, it is well known that $hg \div l$, will express the distance which a heavy body would descend freely from rest upon the inclined plane, in the first second of time. From whence, and by cor. 2, prop. 61, sect. vi, Emerson's Mechanics, $(SC \div SO) \times (hg \div l)$ will express the distance which the cylinder will roll upon the inclined plane in the first second.

Then by the law of uniformly accelerated motion,

$$\frac{SC}{SO} \times \frac{hg}{l} : l :: 1 : t^2; \text{ whence } \frac{SC}{SO} = \frac{2R^2}{3R^2 + r^2} = \frac{l^2}{hgt^2}$$

$$\text{which gives } r = \frac{R\sqrt{(ahgt^2 - 3l^2)}}{l}. \quad Q. E. I.$$

This question was also answered by S. C. I.

XVII. QUESTION 227, by Mr. CUNLIFFE.

Required the sum of the infinite series

$$\frac{1}{1 \cdot 3^2} + \frac{2}{3 \cdot 5} + \frac{3}{5 \cdot 7^2} + \frac{4}{7 \cdot 9^2} + \frac{5}{9 \cdot 11^2} + \&c.$$

SOLUTION, by Mr. J. WALLACE, Edinburgh.

The general term of this series is evidently $\frac{x}{(2x-1)(2x+1)^2}$

And, by applying the common rules for the decomposition of fractions, we find

$$\frac{x}{(2x-1)(2x+1)^2} = \frac{1}{8} \cdot \frac{1}{2x-1} + \frac{1}{4} \cdot \frac{1}{(2x+1)^2} - \frac{1}{8} \cdot \frac{1}{2x+1}.$$

Hence

Hence, putting S for the sum of the given series, we have

$$S = \begin{cases} \frac{1}{8} (1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c. \\ + \frac{1}{8} (-\frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} - \&c. \\ + \frac{1}{4} (\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c. \end{cases}$$

that is, $S = \frac{1}{8} + \frac{1}{4} (\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$

But it is known that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

(Euleri Introd. in Anal. Infin. § 175). Hence we have

$$S = \frac{1}{8} + \frac{1}{4} (\frac{\pi^2}{8} - 1) = \frac{(3 \cdot 1416)^2}{32} - \frac{1}{8} = .1734246.$$

XVIII. QUESTION 228, by Mr. WALLACE.

Find the nature of a curve, such, that any normal shall have a given ratio to the segment of the axis intercepted between it and a given point ?

SOLUTION, by Mr. LOWRY.

Let A , (fig. 149, pl. 7,) be the given point in the axis AC , and BC a normal to the required curve BH . Draw ED perpendicular to AC , and put $AD = x$, and $DB = y$;

then $\dot{x} : \dot{y} :: y : \frac{y\dot{y}}{\dot{x}} = DC$, and $AC = x + \frac{y\dot{y}}{\dot{x}}$.

Also, $BC = \sqrt{(DB^2 + DC^2)} = \sqrt{(y^2 + \frac{y^2\dot{y}^2}{\dot{x}^2})} = y \sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})}$;

therefore, if AC have to BC the given ratio of m to n , we have

$x + \frac{y\dot{y}}{\dot{x}} = \frac{m}{n} y \sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})}$ for the fluxionary equation of the curve.

This equation may be integrated in the same manner as that in the solution to question 218, by first taking the fluxion, supposing $\dot{x}$ or $\dot{y}$ to be constant. Thus, if we suppose $\dot{x}$ to be constant, the

fluxion

fluxion of $x + \frac{y\dot{y}}{\dot{x}}$ is $\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}}$, and the fluxion of $\frac{m}{n} y \sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}} = \frac{m\dot{y}}{n\dot{x} \sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}} (\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}})$;

therefore $\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}} = \frac{m\dot{y}}{n\dot{x} \sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}} (\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}})$;

or $(\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}}) (1 - \frac{m\dot{y}}{n\dot{x} \sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}}) = 0$.

From this equation, we have either $\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}} = 0$, or

$1 - \frac{m\dot{y}}{n\dot{x} \sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}} = 0$. In the first case, by integrating,

we get $x + \frac{y\dot{y}}{\dot{x}} = a$, where a is a constant arbitrary quantity;

therefore $\frac{\dot{y}}{\dot{x}} = \frac{a-x}{y}$, and substituting this value of $\frac{\dot{y}}{\dot{x}}$ in the

original fluxionary equation, it becomes $a = \frac{m}{n} \sqrt{y^2 + (a-x)^2}$,

or $\frac{n^2}{m^2} a^2 = y^2 + (a-x)^2$, an equation to the circle whose ra-

dus is $\frac{n}{m} a$.

As this equation contains a constant arbitrary quantity, it is, of course, the complete integral. We shall now show how the par-

ticular integral is derived from the equation $1 - \frac{m\dot{y}}{n\dot{x} \sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}} = 0$. Here, multiplying by $n\dot{x} \sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}$, and properly re-

ducing the equation, we get $n^2 (\dot{x}^2 + \dot{y}^2) = m^2 \dot{y}^2$, or $\frac{\dot{y}}{\dot{x}} =$

$\frac{n}{\sqrt{m^2 - n^2}}$; and substituting this value of $\frac{\dot{y}}{\dot{x}}$ in the original equation, it becomes

$$x + \frac{yn}{\sqrt{m^2 - n^2}} = \frac{m}{n} y \sqrt{1 + \frac{n^2}{m^2 - n^2}}; \text{ or } x = \frac{y \sqrt{m^2 - n^2}}{n},$$

an equation to a straight line. So that either a straight line, or a circle, will answer the conditions of the problem. This indeed is obvious, for if a triangle ABC be formed, so that BC may be perpendicular to AB, and AC to CB in the given ratio of m to n ; then it is evident that either the straight line AB, which is the locus of the particular integral equation; or any circle described to touch that line, and having its centre on the axis AC, will satisfy the conditions of the problem.

The same otherwise.

Put $z = AC$; then $BC = \frac{n}{m} z$, $y^2 = \frac{n^2}{m^2} z^2 - (z - x)^2$, and
 $y\dot{y} = \frac{n^2}{m^2} z\dot{z} - (z - x)(\dot{z} - \dot{x})$. But $\frac{y\dot{y}}{x} = DC = z - x$;
 therefore $y\dot{y} = (z - x)\dot{x}$, and $(z - x)\dot{x} = \frac{n^2}{m^2} z\dot{z} - (z - x)(\dot{z} - \dot{x})$,
 or $\dot{z}(\frac{m^2 - n^2}{m^2} z - x) = 0$.

From this equation we have $\dot{z} = 0$, or $\frac{m^2 - n^2}{m^2} z - x = 0$.

When $\dot{z}$ is $= 0$, z is $=$ to a constant arbitrary quantity a ;
 and writing a for z in the equation $y^2 = \frac{n^2}{m^2} z^2 - (z - x)^2$, it
 becomes $y^2 = \frac{n^2}{m^2} a^2 - (a - x)^2$,

or $y^2 + (a - x)^2 = \frac{n^2}{m^2} a^2$, the same as before.

Again, when $\frac{m^2 - n^2}{m^2} z - x = 0$, z is $= \frac{m^2}{m^2 - n^2} x$,
 and the equation $y^2 = \frac{n^2}{m^2} z^2 - (z - x)^2$ becomes, after pro-
 per reduction, $x = \frac{y \sqrt{m^2 - n^2}}{n}$.

XIX. QUESTION 229, by Mr. LOWRY.

Two equal bodies, connected by a flexible string, are placed
 on a smooth horizontal table, the string being stretched out in a
 direction

direction parallel to the edge of the table. To the middle of this string, there is fastened another string, at the extremity of which is a given weight that hangs over the edge of the table, in a vertical direction, and, by its descent, communicates motion to the other bodies. Now, supposing the weights of the bodies and the length of the connecting string to be given; it is required to define the motion of the descending weight, and to ascertain its place at the end of any assigned time?

SOLUTION, by Mr. LOWRY, the Proposer.

This question may easily be resolved by the method used by M. D'Alembert, in his *Traite Dynamique*, where the following proposition is assumed as a general principle: "In whatever manner several bodies change their actual motions, if we conceive that the motion which each body would have in the succeeding instant, if it were quite free, is decomposed into two others, of which one is the motion which it really takes in consequence of their mutual action, the second must be such, that if each body were impelled by this force alone, (that is, by the force which produces the second motion,) all the bodies would remain in equilibrio."

Suppose that the bodies A, B are at A' and B' (fig. 150, pl. 7.) when the motion commences, and let D be the middle of the string. Then the point D will be drawn along the straight line DI, perpendicular to A'B', by the action of gravity on the descending weight W; and the equal bodies A, B will describe equal portions of the similar curves A'A, B'B with the same velocities. At the end of any indefinite time t , let the bodies on the table be at A and B, and the middle of the string at D; then the distance DD will be equal to the space through which the weight W has descended in that time. Let the velocity of the bodies in the curves A'A, B'B = u , and their mass, or weight, = w . And the velocity of the descending weight = v , and its mass, or weight, = W . Draw AE perpendicular to D'D; and put $a = AD = A'D =$ half the length of the string, $s = D'D$, $x = D'E$, $y = EA$, $r =$ the arch A'A, $z = ED$, and $g = 32\frac{1}{2}$ feet, the measure of the accelerative force of gravity.

Then, at the end of the time t , when the bodies on the table have acquired the velocity u , if we suppose the weight W to be disengaged from the string, so that the bodies may be at liberty to move freely, it is evident that the bodies A, B would proceed with the uniform velocity u ; and in the succeeding instant t the descending body W, would acquire the velocity $v + gt$, and

the quantities of motion would then be

$$wu, wu \text{ and } W(v + g\dot{t});$$

or $w(u + \dot{u} - \dot{u})$, $w(u + \dot{u} - \dot{u})$ and $W(v + \dot{v} + g\dot{t} - \dot{v})$.

But the motions which the bodies really take at that time, are $w(u + \dot{u})$, $w(u + \dot{u})$, and $W(v + \dot{v})$; therefore the motions

$w(-\dot{u})$, $w(-\dot{u})$, and $W(g\dot{t} - \dot{v})$ must be such that if the bodies were impressed with these motions alone, the system would be in equilibrio;

$$\text{that is, } 2w(-\dot{u}) \text{ must be } = \frac{v}{u} W(g\dot{t} - \dot{v});$$

or $Wg\dot{t} = W\dot{v} + 2w\frac{u\dot{u}}{v}$. But $\dot{t}$ is $= \frac{\dot{s}}{v}$, therefore

$$gWs = Wv\dot{v} + 2w\dot{u}u; \text{ and taking the fluents, we get } 2gWs = Wv^2 + 2wu^2 \dots\dots\dots (1)$$

But it is evident that the string AD is a tangent to the curve at A, therefore $\dot{r} : \dot{x} :: a : z$, or $\dot{r} = \frac{a\dot{x}}{z}$. Also $\dot{x} : -\dot{y} :: z : y$,

or $-\dot{y} = \frac{y\dot{x}}{z}$; and since $a^2 - y^2$ is $= z^2$, $-\dot{y}$ is $= \frac{z\dot{z}}{y}$;

therefore $\frac{y\dot{x}}{z}$ is $= \frac{z\dot{z}}{y}$, $\dot{z} = \frac{y^2\dot{x}}{z^2}$ and $\dot{x} = \frac{z^2\dot{z}}{a^2 - z^2}$; conse-

quently $\dot{s} = \dot{x} + \dot{z} = \dot{x} + \frac{y^2\dot{x}}{z^2} = \frac{a^2\dot{x}}{z^2} = \frac{a^2\dot{z}}{a^2 - z^2}$; $s = \int \frac{a^2\dot{z}}{a^2 - z^2}$,

$$= a \times \log. \frac{a+z}{a-z}, \text{ and } \frac{v}{u} = \frac{\dot{s}}{\dot{r}} = \frac{z}{a}, \text{ or } u^2 = \frac{a^2v^2}{z^2}.$$

This value of u^2 being substituted in the equation (1), it becomes $2gWs = Wv^2 + 2w\frac{a^2v^2}{z^2}$; from whence we deduce

$$v^2 = \frac{2gWsz^2}{Wz^2 + 2wa^2} = \frac{2gWa \log. \frac{a+z}{a-z}}{Wz^2 + 2wa^2}; \text{ and}$$

$$t = \int \dot{s} \sqrt{\left(\frac{Wz^2 + 2wa^2}{2gWsz^2}\right)} = \int \frac{a^2\dot{z}}{a^2 - z^2} \sqrt{\left(\frac{Wz^2 + 2wa^2}{2gWz^2 a \log. \frac{a+z}{a-z}}\right)}$$

In

In the particular case when $z = a$, we have

$$v^2 = \frac{2gW}{W + 2w}, \text{ and } t = s \sqrt{\left(\frac{W + 2w}{2gW} \right)} \text{ as they ought to be.}$$

To find the equation of the curve described by the bodies, we have

$$\dot{x} = \frac{z^2 \dot{z}}{a^2 - z^2} = \frac{\dot{y}}{y} \sqrt{(a^2 - y^2)}, \text{ and}$$

$$x = a \times \log. \frac{a + \sqrt{(a^2 - y^2)}}{y} - \sqrt{(a^2 - y^2)}.$$

XX. QUESTION 230, by Mr. LOWRY.

A circle and a sphere being given by position, it is required to determine the perspective representation of the circle on the surface of the sphere, and to find the rectification of the curve; supposing the eye to be at the centre of the sphere?

SOLUTION, by Mr. LOWRY, the Proposer.

Let C (fig. 151, pl. 7.) be the centre of the sphere, PQR the circle given by position, and O its centre. Draw CG perpendicular to the plane of the circle; then G is a given point, and CG a given line. Draw GO to meet the circle in Q and R, and let P be a point in the periphery of the circle; then if PC be drawn to meet the surface of the sphere in p , it is evident that the point p is in the required curve. To find the position of this point on the surface of the sphere, let GP, Gp be drawn. Then since CG is perpendicular to the plane GPR, the angle CGP is a right-angle, therefore if we make a right-angled triangle CGP, and take Cp = to the radius of the sphere, the distances GP, Pp will be known: and if with these distances and the centres G, P we describe arches on the surface of the sphere, they will intersect in p . In this manner we may find as many points on the surface of the sphere as we please, and the curve described through these points will be the representation of the given circle.

If we conceive the curve to be projected on the plane which passes through the points C, G, O, we may easily describe the projected curve. For if pm be drawn perpendicular to the plane CGO, the point m will be in the curve. Draw PM perpendicular, and mg parallel to RQ, and join p, g . Then it is evident that the triangles GFM, gpm are similar, therefore PC :

pC

$pC = r' :: PM : pm :: GM : gm :: CG : Cg$, and the rectangular co-ordinates gC, gm are equal to $r' \times \frac{CG}{PC}$ and $r' \times \frac{GM}{PC}$ respectively.

The nature of the curve may also be defined by equations as follows: put $x = Cg, y = gm, z = pm, d = GO, h = CG$, and $r = OP$; then $x : y :: h : GM = \frac{yh}{x}$; therefore $OM = \frac{yh}{x} - d$, and $PM^2 = r^2 - (\frac{yh}{x} - d)^2$.

Also $h^2 : x^2 :: r^2 - (\frac{yh}{x} - d)^2 : z^2$,

$$\text{or } z^2 = \frac{x^2}{h^2} \{ r^2 - (\frac{yh}{x} - d)^2 \} \dots\dots\dots (1)$$

And, by the property of the sphere,

$$z^2 + y^2 + x^2 = r^2. \dots\dots\dots (2)$$

These equations serve to define the nature of the curve on the surface of the sphere; and if we eliminate z , we get

$$\frac{x^2}{h^2} \{ r^2 - (\frac{yh}{x} - d)^2 \} = r^2 - x^2 - y^2, \text{ for the}$$

equation of the curve when projected on the plane OGC . By reducing this equation it becomes $(r^2 + h^2 - d^2) x^2 - h^2 r^2 + 2dxyh = 0$, which is an equation to the common

hyperbola; and when d is $= 0$, x is $= \frac{h^2 r^2}{\sqrt{(r^2 + h^2)}}$, and the projection of the curve is then a straight line.

Again, if x be eliminated from the equations (1), (2), we have

$$\sqrt{(r^2 - y^2 - z^2)} = \frac{dhy}{r^2 - d^2} + \frac{hr}{r^2 - d^2} \sqrt{(y^2 - z^2 - \frac{d^2 z^2}{r^2})}$$

for the equation of the curve when projected on the plane GPR . When d is $= 0$, $(y^2 + z^2)(h^2 + r^2)$ is $= r^2 r^2$, an equation to a circle.

From the above equations we may find the fluxions of x, y , and z , in terms of one of these variables and its fluxion, and then $\int \sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}$ will express the length of the curve: but the operation will be much simpler if we proceed in the following manner: If the circle PRQ be considered as the base, and the point C as the vertex of a cone, it is obvious that the curve in the question is formed by the mutual intersection of the cone and

and spherical superficies; and if we conceive the surface of the cone to be extended on a plane, (fig. 152, pl. 7.) and the arch EF described about the centre C, with a distance equal to the radius of the sphere, it is evident that the arch EF is the development of the curve on the plane, and consequently its length is equal to the length of the curve.

Let w be any variable arch PR, and PP' a small part of the arch. Draw PC, P'C to meet the arch EF in p and n , and draw PN perpendicular to CP'. Let ϕ be the measure of the arch w on the circle having unity for its radius; put u = the arch Ep, and p = CP. Then by similar triangles $u = p\pi : PN$

$$\therefore Cp : CP, \text{ or } u = \frac{r'}{p} \times PN. \text{ But } PN \text{ is } = \sqrt{(P'P^2 - P'N^2)} \\ = \sqrt{(\dot{w}^2 - \dot{p}^2)}, \text{ therefore } u = \frac{r'}{p} \sqrt{(\dot{w}^2 - \dot{p}^2)}.$$

Now it is evident that $p^2 = h^2 + d^2 + r^2 - 2rd \cos \phi$, hence $\dot{p} = r\dot{\phi} \times \frac{d \sin \phi}{p}$. But $\dot{w} = r\dot{\phi}$; therefore, by substitution,

$$u = \frac{r'r\dot{\phi}}{p} \sqrt{\left(1 - \frac{2d \sin^2 \phi}{p}\right)} = r'r\dot{\phi} \sqrt{\frac{\{h^2 + (r - d \cos \phi)^2\}}{h^2 + d^2 + r^2 - 2rd \cos \phi}},$$

and the length of the curve

$$= r'r\dot{\phi} \sqrt{\frac{\{h^2 + (r - d \cos \phi)^2\}}{h^2 + d^2 + r^2 - 2rd \cos \phi}}.$$

For the method of integrating this expression, we shall refer to Art. 54, *Memoire sur les Transcendental Elliptiques*, par Le Gendre.

It is obvious that most of the properties noticed in the solutions to questions 220 and 231, will be true for the curve in the question, as well as for a small circle.

XXI. QUESTION 231, by Mr. LOWRY.

If from two given points on the surface of a sphere, two great circles be drawn to any point D in the circumference of a small circle given by position, meeting it again in E and F: then the great circle joining the points E, F will either pass through a given point P; or a great circle drawn through E and P will cut off from the small circle EDF, an arch FG of a given length.

SOLUTION,

SOLUTION, by Mr. LOWRY.

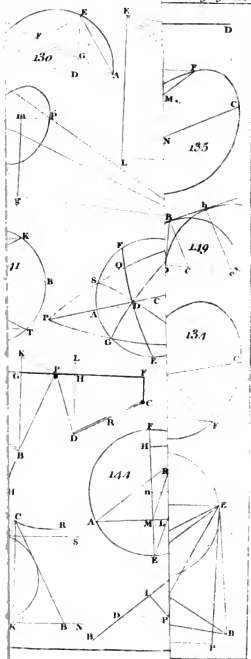
Let A, B (fig. 153, pl. 8.) be the given points on the surface of the sphere, and let the planes of the great circles AED, BFD, AB and GPE meet the plane of the small circle in the straight lines aED , bFD , ab and GpE ; then it is evident that a and b are given points.

Also, let the straight line drawn from P, to the centre O of the sphere, meet the line GE in p ; and through p , and the centre C of the given circle, draw a straight line to meet ab in I, and the circle in L and M; then, since (by hyp.) P is a given point, p will also be a given point; and because the arch EF either passes through P, or the arch FG is of a given length, the line EF must either pass through p , or the angle FEp must be a given angle; therefore the point p must be so situated, that straight lines being drawn from the points a , b to any point D in the periphery of the circle, meeting it again in E and F, the line EF may either pass through p , or make a given angle with a straight line passing through that point. This we know (from the solution to quest. 210) will be the case when the rectangle $aI \times ab$ is equal to the square of the tangent aH drawn to the circle from a ; and when the line IM is harmonically divided in the points I, L, p , M; or when $IC \times Cp = CL^2$. For then the angle FEp is equal to the difference between the given angle bIC and a right-angle; and when bIC is a right-angle, the line EF will pass through p .

Let the plane which passes through the points I, O, C meet the surface of the sphere in the great circle QLPM, and the plane bOa in the straight line IQO. Also let bK be drawn to touch the circle at K, and through the points A, H; and B, K let arches of great circles be drawn; and join O, H and O, K. Then since the planes OaH , ObK touch the small circle at H and K, the arches AH, BK will also touch the circle in these points, and are therefore given arches; and it is evident that aH is the tangent, and aO the secant, of the arch AH; and bK the tangent, and bO the secant, of the arch BK.

Now take $A'B'$ = the given arch AB, and make $A'O'$, $B'O'$ each equal to the radius of the sphere. Take $O'a'$ = to the secant of the arch AH, and $O'b'$ = to the secant of the arch BK, and join $a'b'$. Also make the rectangle $Ia' \times a'b' = \tan^2 AH$, and draw IO' meeting the arch $A'B'$ in Q. Then the arch $A'Q'$ is evidently equal to the arch AQ.

Again, since the line IM is harmonically divided in the points I, L, p , M;





I, L, p , M; the arch QM is also harmonically divided in the points Q, L, P, M (solution to quest. 220); or which is the same thing, $\tan. QS \tan. PS = \tan.^2 SL$, S being the pole of the circle EDF.

Hence, if on the arch AB, we take AQ = to the arch A'Q', and the arch SQ be divided in P, so that $\tan. QS \tan. PS$ may be equal $\tan.^2 SL$, the point P will be that required.

In the figure to this solution, the given points A, B are taken without the circle EDF; but the same reasoning will apply when they are within the circle; or when one is within, and the other without; only, when they are within the circle, the lines aH , bK must be drawn perpendicular to the diameters which pass through the points a , b ; and the distances aH , aO , bK and bO are the sines and cosines of the arches AH, BK instead of their tangents and secants.

It is evident (from the solution to quest. 210, and what has been done above,) that the proposition may be extended to any number of given points A, B, C, D, &c.; and it is also obvious from thence how a great circle spherical polygon may be inscribed in a given small circle, so that each side may pass through a given point on the surface of the sphere.

SCHOLIUM. We have already remarked that, when the line IC is at right angles to ab , the line EF will pass through the point p . In this case the sum of the squares of aH and bK is equal to the square of ab ; and if the line aF meet the circle again in N, the points b , N, E are in a straight line; or if the opposite sides ED, FN of a trapezium inscribed in the circle EDF, intersect in a , the sides FD, EN will intersect in b . Also the diagonals DN, EF will intersect in the given point p . These properties are generally known when the figure is described on a plane (see solution to quest. 240, Old Series of the Repository), and it is obvious that there are similar properties when the figure is on the surface of a sphere. Thus, if the arch AF meet the small circle in N, the points B, N, E are in the arch of a great circle; or if the opposite sides of any great circle trapezium, inscribed in the small circle EDF, intersect in A, the other opposite sides will intersect in B. Also the great circles joining the points E, F and D, N will intersect in the given point P.

Again, let r be the radius of the sphere, then by trigonometry

$$ab^2 = aO^2 + Ob^2 - 2aO \times Ob \times \cos aOb$$

$$= \sec^2 AH + \sec^2 BK - \frac{2 \sec AH \sec BK \cos AB}{r},$$

and $aH^2 + bK^2 = \tan^2 AH + \tan^2 BK = \sec^2 AH + \sec^2 BK - 2r^2$;

therefore, when ab^2 is $= aH^2 + bK^2$,

$$\frac{\sec AH \sec BK \cos AB}{r} = r^2$$

$$\text{or, } r \cos AB = \frac{r^4}{\sec AH \sec BK} = \cos AH \cos BK.$$

From whence we have the following properties.

1st. Let EDF (fig. 154, pl. 8.) be a small circle given by position, and A and B two given points on the surface of a sphere, such, that the rectangle of the radius of the sphere, and the cosine of AB, may be equal to the rectangle of the cosines of the tangents AH, BK drawn to the circle from A and B, and let great circles be drawn from the points A and B to any point D in the periphery of the circle EDF, meeting it again in E and F: then the arches joining the points A, F and B, E will intersect in the periphery of the circle EDF. Also the arches joining the points E, F and D, N will intersect in a given point P.

2d. If the opposite sides of a spherical trapezium inscribed in a small circle of the sphere, be produced to meet; the rectangle of the radius of the sphere, and the cosine of the arch joining the points of concurrence, will be equal to the rectangle of the cosines of the tangents drawn to the circle from these points.

XXII. QUESTION 232, by Mr. WALLACE.

Let F be the focus of a conic section, and P a given point in the axis. Draw FA to any point of the curve; take PB in the axis equal to FA and join AB. There is a straight line given by position, which divides AB into parts having to each other a given ratio. Required the demonstration?

FIRST SOLUTION, by Mr. LOWRY.

Suppose that LI (fig. 155, pl. 8.) is the straight line given by position which divides AB into parts (AI, IB) having a given ratio.

Let AE, drawn parallel to the axis, meet PI produced in E; then, by similar triangles, AI is to IB as AE to BP, or AF; but AI has to IB a given ratio by hypothesis; therefore AE has to AF a given ratio; and, consequently, the point E is in the directrix of the conic section. Let the directrix ED be drawn to meet the axis in D, and let V be the vertex of the axis. Then since EI has to IP the given ratio of AE to AF (PB); that is, of DV to VF; and the point E being always in the

the directrix ED, the point I will be in a straight line LI, given by position, and parallel to ED; wherefore DL is to LP as DV to VF. Hence, if DP be divided at L, in the given ratio of DV to VF, the point L will be given, and the straight line LI, which divides AB in a given ratio, will be given by position.

The proposition, however, is not confined to the particular case where PB is taken on the axis, but is true when PB is taken on any straight line given by position.

For, if P (fig. 156, pl. 8.) be a given point in a straight line PQ given by position, (which meets the curve at V,) and BP be taken $\equiv$ FA; there is a straight line LI, given by position, which divides AB into parts having a given ratio. Let AE, drawn parallel to PQ, meet PI produced in E; then it may be shewn in the very same manner as above, that the point E is in the directrix of the conic section; and that DL is to LP as DV to VF; therefore L will be a given point, and the line LI will be given by position, and AB will be divided at I in the given ratio of DV to VF.

SECOND SOLUTION, by Mr. CUNLIFFE.

Let VAF (fig. 155, pl. 8.) be a given conic section, F its focus, V its principal vertex, DE its directrix at right angles to DVF. Draw AE parallel to DF meeting DE in E, and draw PE intersecting AB in I. Lastly, through I draw LM parallel to DE, meeting DF in L and EA in M.

By the known property of the directrix $FA = PB : EA :: FV : VD$, a given ratio. Also, per the similar triangles, EIA, PIB, and the parallels LI, DE,
 $PB : EA :: IB : IA :: PI : IE :: PL : LD$; therefore
 $IB : IA :: PL : LD :: FV : VD$, a given ratio. Wherefore divide PD in L in the given ratio of FV to VD, and draw LM parallel to the directrix; then LM will be a right line given by position, which always divides AB in a given ratio.

XXIII. QUESTION 233.

To this question we have received no answer.

XXIV. QUESTION 234, by Mr. IVORY.

A point may be found within the base of every oblique cone, such, that if any plane be drawn through the vertex and the point found to cut the cone, then the vertical angle of the
a H a
triangle

triangle, which is the common section of the plane and the cone will be bisected by the right line drawn from the point found to the vertex of the cone?

SOLUTION, *by Mr. LOWRY.*

LEMMA. Let a straight line EI (fig. 157, pl. 8.) be harmonically divided in the points I, F, P, E; and from the points of division let straight lines be drawn to any point V, so that VI may be perpendicular to PV; then the angle EVF is bisected by the line PV.

For through P, draw GPH parallel to IV, meeting VF produced in H; then by similar triangles

$$FI : PF :: IV : PH,$$

$$\text{and } PE : IE :: PG : IV;$$

but, by hypothesis, $IF : IE :: PF : PE;$

therefore, ex æquo, $IV : PH :: IV : PG;$

therefore PH is equal to PG, and the right-angled triangles PGH, PHV are equal in every respect; consequently the angles GVP, HVP are equal, or the angle EVF is bisected by the line PV.

Now let a plane be drawn through the axis VC (fig. 158, pl. 8.) of the cone, perpendicular to the plane of the base, meeting the surface of the cone in the straight lines AV, BV. Draw VP to bisect the angle AVB; then P is the point required. That is to say, if any plane be drawn through PV, to meet the surface of the cone in the straight lines EV, FV, then the angle EVF will be bisected by the line PV.

Let a plane be drawn through V, perpendicular to the line VP, meeting the plane of the base in the straight line HI, and the diameter AB produced in D; and join VD. Then VD is perpendicular to VP (Euc. XI. 17.) and IDH perpendicular to AD (XI. 18.). But since PV bisects the angle AVB, and VD is perpendicular to PV, it is evident that VD bisects the exterior angle BVa; therefore $AP : PB :: AV : BV :: AD : BD$ (VI. 3. A); or the diameter is harmonically divided in the points D, B, P, A.

Let DF be produced to meet DH in I, and join VI. Then because the diameter of the circle is harmonically divided in the points D, B, P and A; and the line HDI is perpendicular to AD; any line EI drawn through P to meet the circle in E and F, and the straight line DI in I, is harmonically divided in the points I, F, P and E. This property of the circle is generally known. Moreover the plane DIV being perpendicular to VP, the line IV is perpendicular to PV; therefore, by the lemma, the angle EVF is bisected by the line PV.

When

When EF is parallel to DI, it is obvious that the angles FPV, EPV are right-angles, and that EP is equal to PF; therefore, in this case, the triangles EPV, FPV are equal in every respect; consequently the line PV bisects the angle EVF.

Corol. The rectangle $EV \times VF$ is equal to the constant rectangle $AV \times VB$.

For $EV \times VF = VP^2 + EP \times PF = VP^2 + AP \times PB = AV \times VB$.
(VI. Prop. B.)

XXV. QUESTION 235, by Mr. IVORY.

It is required to shew that the surface of a right cone on an elliptical base may be found by the rectification of the ellipse.

By a right cone on an elliptical base is meant such a cone as has an ellipse for its base, and likewise the right line, drawn from the vertex to the centre of the ellipse, perpendicular to the plane of the base?

SOLUTION, by Mr. LOWRY.

Let V (fig. 159, pl. 8.) be the vertex, C the centre of the base, CB the semi-transverse, and CD the semi-conjugate axis. Let $DH = z$ be a variable arch of the elliptic base, and Hh an indefinitely small arch $= \dot{z}$. Join VH, Vh and CH; and draw Hl perpendicular to CB. Put $h = CV$, the height of the cone; e = the eccentricity of the ellipse, the semi-transverse being 1; $r = VH$, the distance of the point H from the vertex $x = CI$ and $y = IH$.

From h draw a line perpendicular to VH; then this perpendicular will be one leg of a right-angled triangle, of which $\dot{z}$ is the hypotenuse, and $\dot{r}$ the other leg; it will therefore be equal to $\sqrt{(\dot{z}^2 + \dot{r}^2)}$; and the fluxion of the conical surface DVH, or the area of the small triangle HVh, will be equal to $\frac{1}{2}r \sqrt{(\dot{z}^2 + \dot{r}^2)}$. Therefore, if s be put for the surface of the cone we have $\dot{s} = \frac{1}{2}r \sqrt{(\dot{z}^2 + \dot{r}^2)}$.

The equation of the ellipse being $y^2 = (1 - e^2)(1 - x^2)$, we have $CH^2 = x^2 + y^2 = x^2 + (1 - e^2)(1 - x^2) = 1 - e^2 + e^2x^2$; therefore $r^2 = CV^2 + CH^2 = h^2 + 1 - e^2 + e^2x^2$, and $\dot{r} = e^2x\dot{x} \div r$.

Also $\dot{z}$ is $= \dot{x} \sqrt{\frac{1 - e^2x^2}{1 - x^2}}$; therefore, by substitution,

$$\dot{s} =$$

$\dot{s} = \frac{r}{2} \dot{x} \sqrt{\left(\frac{1 - e^2 x^2}{1 - x^2} + \frac{e^4 x^2}{r^2} \right)}$; and putting for r its value $\sqrt{(h^2 + 1 - e^2 + e^2 x^2)}$, and reducing the quantity under the radical, we get

$$\dot{s} = \frac{1}{2} \dot{x} \sqrt{\frac{h^2 + 1 - e^2 + h^2 e^2 x^2}{1 - x^2}}.$$

But since $1 - e^2$ is equal to the square of CD , $h^2 + 1 - e^2$ is equal to the square of VD , the line drawn from the vertex to the extremity of the conjugate axis; therefore, if d be put for that line, we have

$$\frac{1}{2} d \dot{x} \sqrt{\left(\frac{1 - \frac{h^2 e^2}{d^2} x^2}{1 - x^2} \right)} \text{ for the fluxion of the conical}$$

surface; the fluent of which is obviously $\frac{1}{2} d$ multiplied by the periphery of the ellipse whose excentricity is $(h^2 \div d^2) e^2$, and semi-transverse 1.

The same otherwise.

Fig. 160, pl. 8. Let a plane, parallel to the plane of the base, be drawn through any point E , in the line VC , to meet the plane CVB in the straight line EQ . Draw FG perpendicular to EQ meeting the surface of the cone in G , and VF . VG to meet the plane of the base in I and H ; then the point H will be in the periphery of the base, and HI will be perpendicular to CB . Put $y = VE$, $x = EF$, $z = FG$, and $s =$ the conical surface bounded by the plane EQG ; then,

$$s = \iint \dot{x} \dot{y} \sqrt{\left(1 + \frac{\dot{z}^2}{x^2} + \frac{\dot{y}^2}{y^2} \right)}.$$

Put $h = VC$, $u = CI$ and $v = IH$; then by similar triangles $y : x :: h : u = hx \div y$, and $y : z :: h : v = hz \div y$;

$$\text{hence } x = \frac{uy}{h} \text{ and } z = \frac{vy}{h};$$

Take fluxions, supposing y constant; then

$$\dot{x} = \frac{y}{h} \dot{u}, \text{ and } \dot{z} = \frac{v}{h} \dot{v}; \text{ therefore } \frac{\dot{z}}{x} = \frac{\dot{v}}{u}.$$

Again, $\dot{z} = \frac{v \dot{y}}{h} + \frac{y \dot{v}}{h}$, and $\frac{\dot{z}}{y} = \frac{v}{h} + \frac{y \dot{v}}{y h}$; but $\dot{y}$ is $= -y \frac{\dot{u}}{u}$, when x is constant; therefore

* For the investigation of this formula, see *Traité Élémentaire de calcul def. et integ. par La Croix.* Art. 251.

$$\frac{\dot{z}}{y} = \frac{v}{h} - \frac{u\dot{v}}{h\dot{u}}.$$

Whence, by substitution,

$$s = -\frac{1}{h^2} \iint y \dot{y} \dot{u} \sqrt{\left\{ h^2 + h^2 \left(\frac{\dot{v}}{\dot{u}} \right)^2 + \left(v - \frac{u\dot{v}}{\dot{u}} \right)^2 \right\}}.$$

The fluent relative to y , being taken from $y = 0$ to $y = h$,

we have $s = \frac{1}{2} \int \dot{u} \sqrt{\left\{ h^2 + h^2 \left(\frac{\dot{v}}{\dot{u}} \right)^2 + \left(v - u \frac{\dot{v}}{\dot{u}} \right)^2 \right\}}$ for the surface of the whole cone.

This is a general expression for the surface of a cone, let the figure of the base be what it may: In the present case $v^2 = (1 - e^2)(1 - u^2)$; therefore

$$v = \sqrt{(1 - e^2)(1 - u^2)}, \quad \frac{\dot{v}}{\dot{u}} = -u \sqrt{\frac{1 - e^2}{1 - u^2}}, \text{ and } \left(v - u \frac{\dot{v}}{\dot{u}} \right)^2 = \frac{1 - e^2}{1 - u^2};$$

therefore, by substitution, and proper reduction, we get

$$s = \frac{1}{2} \int \dot{u} \sqrt{\left(\frac{h^2 + 1 - e^2 - h^2 e^2 u^2}{1 - u^2} \right)}, \text{ the same expression}$$

as before.

XXVI. QUESTION 236, by Mr. IVORY.

From a point assumed without a cone, let two tangents be drawn to the same conic surface, or to the opposite surfaces; and let a plane be drawn through the vertex of the cone and the two points of contact of the tangents: then, if a right line be drawn from the assumed point to cut the plane, and either of the conic surfaces in two points, that right line will be harmonically divided?

N. B. The demonstration of some properties of all the conic sections readily follows from this proposition.

SOLUTION, by Mr. LOWRY.

Let P (fig. 161, pl. 8.) be the assumed point, PQ and PR the tangents; and let the plane drawn through the vertex, and the points of contact P and Q, cut the conic surface in the lines VQE and VRF, and the plane drawn through P, parallel to the plane of the base, in the line EF. Then any straight line drawn from P to meet the plane VEF in I, and either of the conic surfaces in H and G, is harmonically divided in the points P, H, I and G. For let a plane be drawn through the vertex V and

and the line PG to cut the conic surface in the lines VGA and VHB, the plane VEF in the line VD, and the plane of the circle in the line ADBP; also draw PE and PF. Then since the lines PQ and PR touch the conic surface, or the opposite surfaces, at Q and R, the planes VQEP and VRFP will touch the conic surface in the lines VE and VF, and consequently PE and PF will touch the circle AEBF at E and F; therefore (by a well known property of the circle) PA is harmonically divided in the points P, B, D and A; therefore the lines PV, BV, DV and AV are harmonicals, and since PG meets these lines in the points P, H, I and G, it is harmonically divided in these points, (Part 2nd. Lem. 1. B. V. Hamilton).

If PG be parallel to VA, or to VB, then PI is bisected in H, or in G, (Part 1st. Ibid.)

COR. 1. A line drawn through any point in VP to meet the plane VEF, and either of the conic surfaces, is harmonically divided. For it will meet four harmonicals.

COR. 2. If GH be bisected in K; then (by reason of the harmonical division) $KI \times KP$ is equal to KH^2 or KG^2 .

It follows from this proposition, that if two tangents be drawn from the same point to any conic section, then any line drawn through P to meet the section in H and G, and the line RQ that joins the points of contact in I, is harmonically divided in the points P, H, I and G.

If the assumed point be within the cone, we have the following proposition, which is analogous to the preceding one.

Through any assumed point P within a cone, (fig. 162, pl. 8.) let a plane be drawn parallel to the plane of the base, cutting the conic surface in the circular section ABL. Let C be the centre of the circle, and on CP let the point Q be taken such, that CQ may be a third proportional to CP, and the radius CL, and let EF be drawn perpendicular to CQ. Then any right-line drawn through P to meet the conic surface in G and H, and the plane VEF in I, is harmonically divided in the points G, P, H and I.

For, let the plane drawn through V, and the line GPI, cut the conic surface in the lines VAG and VHB, the plane of the circle in the line APBD, and the plane VEF in the line VID. Also let PV be drawn. Then because CL, the radius of the circle, is a mean proportional between CP and CQ, and EF is perpendicular to CQ, therefore (by a known property of the circle) the line DA is harmonically divided in the points A, P, B and D; therefore the lines GAV, PV, BV and DV are harmonicals, and consequently the line GI is harmonically divided in the points G, P, H and I.

If GPI be parallel to VA, or to VB, then PI is bisected in H, or in G; and if it be parallel to the plane VEF, then GH is bisected in P.

COR. 1. A line drawn through any point in VD to meet the plane VEF, and the conic surface, is harmonically divided.

COR. 2. If GH be bisected in K, then (by reason of the harmonical division) $KP \times KI$ is equal to KH^2 or KG^2 .

COR. 3. If any plane MN be drawn through P to meet the plane VEF in the right-line IO, and from any point I in this line, tangents be drawn to the conic surface at M and N, the line MN that joins the points of contact, will pass through P. For let IP be drawn to meet the conic surface in H and G, and if MN does not pass through P, let it meet IG in p : then, by what has been already proved, IG will be harmonically divided in the points I, H, P and G, and also in the points I, H, p and G, which is absurd; therefore MN passes through P.

COR. 4. When MN is parallel to IO, it is bisected in P.

From what has been demonstrated it follows:

That, if on the diameter GH of a conic section, two points P and I be assumed such, that $PK \times KI$ may be equal to the square of the semi-diameter KH (or in the parabola that PH may be equal to HI), and IO be drawn parallel to the line MN that joins the points of contact of tangents drawn to the section from I: then any right-line drawn through P to meet the conic section in R and S, and the line IO in T, is harmonically divided in the points R, S, P and T.

Dr. O'RIORDAN answered this question.

XXVII. QUESTION 237, by Mr. IVORY.

It is required to determine the position of the axes of the representation of a circle given by position in a horizontal plane: supposing the place of the eye to be given, and the plane of the picture perpendicular to the plane of the horizon?

SOLUTION, by Mr. LOWRY.

LEMMA 1. Let a cone be cut by a plane, PQ (fig. 163, pl. 8.), which neither passes through the vertex, nor is parallel
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the original circle, O its centre, MN the directing line, and VZ the director of the eye. Draw OD perpendicular to MN , take OE a third proportional to OD , and the radius of the circle; and DL a mean proportional between OD and ED . Bisect the line LV in U , and draw UY perpendicular to VL , meeting MN in Y ; with the radius YL , or YV , and centre Y , describe a circle intersecting MN in I and K . Draw the lines IWT and $KHEC$; then it is evident from the second lemma that, the image of the point E , is the centre C , and that the images of the chords TW and GH , are the axes PQ and RS , of the ellipse, which is the representation of the given circle on the plane of the picture.

Let mn (fig. 166, pl. 8.) be the vanishing line of the plane of the circle, d the centre of that line, and Vd its distance. Also let ab be the representation of the diameter AB , which is perpendicular to the directing line, (its vanishing point being d), and a the representation of the centre of the circle. Take dl a mean proportional between do and dC , and join VI . Bisect VI in u , and draw uy perpendicular to VI , meeting mn in y ; with the radius yl , or yV , describe a circle intersecting mn in i and k , and join i, l , and l, k . Through the centre C , draw γCP and vCR parallel to il and lk respectively, and they will represent the indefinite axes sought. This will appear evident when the corollaries to the third lemma are attentively considered.

To find the lengths of the axes.

Through a , the extremity of the diameter ba , draw γav parallel to the vanishing line mn , meeting the axes in γ and v , and draw $\alpha\gamma$ and $a\beta$ perpendicular to $C\gamma$ and Cv . Take CQ a mean proportional between γC and αC , and CS a mean proportional between vC and βC ; then CQ and CS are the semi-axes sought.

For since a tangent to the circle at the point A is parallel to mn , its image will also be parallel to mn , and a tangent to the ellipse at a ; therefore γav is a tangent to the curve at a ; and by a well known property of the ellipse, the squares of the semi-diameters CQ and CS , are equal to the rectangles $\gamma C \times C\alpha$ and $vC \times C\beta$, which they are by construction, therefore CQ and CS are the semi-axes required.

It is obvious that the image of the given circle will be an ellipse, a parabola, or an hyperbola, according as the circle falls without, touches, or is cut by the directing line. It is the first of these cases only that we have considered in the above solution. A complete solution, however, of all the cases may be found in Hamilton's Perspective, B. III,



XXVIII. QUESTION 238, Mr. IVORY.

Let n be any whole positive number, and put

$$Q = \frac{2 \cdot 4 \cdot 6 \cdot 8 \dots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \dots (2n+1)}; \text{ then the whole value of the}$$

triple fluent $\int \frac{dx}{\sqrt{1-x^2}} \cdot \int \frac{dx}{\sqrt{1-x^2}} \int \frac{x^{2n+1} dx}{\sqrt{1-x^2}}$, which is generated while x increases from 0 to 1, is equal to

$$Q \times \left\{ \frac{1}{(2n+3)^2} + \frac{1}{(2n+5)^2} + \frac{1}{(2n+7)^2} + \frac{1}{(2n+9)^2} + \&c. \right\}$$

N.B. Observe that the case $n=0$ is included by taking $Q = 1$.

FIRST SOLUTION, by Mr. LOWRY.

It is well known that

$$\int \frac{x \dot{x}}{\sqrt{1-x^2}} = 1 - \sqrt{1-x^2}$$

$$\int \frac{x^3 \dot{x}}{\sqrt{1-x^2}} = \frac{2}{3} - \frac{2}{3} (1 + \frac{1}{2} x^2) \sqrt{1-x^2}$$

$$\int \frac{x^5 \dot{x}}{\sqrt{1-x^2}} = \frac{2 \cdot 4}{3 \cdot 5} - \frac{2 \cdot 4}{3 \cdot 5} (1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4) \sqrt{1-x^2}$$

&c. &c.

and in general, that $\int \frac{x^{2n+1} \dot{x}}{\sqrt{1-x^2}} =$

$$Q - Q (1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^6 + \dots \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} x^{2n}) \sqrt{1-x^2};$$

where the first term of each expression is the fluent generated while x increases from 0 to 1.

Therefore the double fluent $\int \frac{\dot{x}}{\sqrt{1-x^2}} \cdot \int \frac{x^{2n+1} \dot{x}}{\sqrt{1-x^2}}$ is =

$$Q \int \left(\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{1}{2} x^2 \dot{x} - \frac{1 \cdot 3}{2 \cdot 4} x^4 \dot{x} - \dots - \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} x^{2n} \dot{x} \right)$$

$$= Q (A - x - \frac{1}{2 \cdot 3} x^3 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 - \dots - \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n+1} x^{2n+1}),$$

A being the circular arc whose sine is x . And if R be put for the

the triple fluent $\int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n+1} \dot{x}}{\sqrt{(1-x^2)}}$ we have

$$R = Q \int \frac{\dot{x}}{\sqrt{(1-x^2)}} (A - x - \frac{1}{2 \cdot 3} x^3 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 - \dots - \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n+1} x^{2n+1}).$$

But $\frac{\dot{x}}{\sqrt{(1-x^2)}}$ being $= \dot{A}$, the fluent of the first term $\frac{A \dot{x}}{\sqrt{(1-x^2)}}$ is $= \frac{A^2}{2}$;

and the fluents $\int \frac{x \dot{x}}{\sqrt{(1-x^2)}}$, $\int \frac{x^3 \dot{x}}{\sqrt{(1-x^2)}}$, $\int \frac{x^5 \dot{x}}{\sqrt{(1-x^2)}}$, to $\frac{x^{2n+1} \dot{x}}{\sqrt{(1-x^2)}}$, generated while x increases from 0 to 1, being

$1, \frac{2}{3}, \frac{2 \cdot 4}{3 \cdot 5}, \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}$, to $\frac{2 \cdot 4 \cdot 6 \dots 2n}{3 \cdot 5 \cdot 7 \dots 2n+1}$ respectively; we

have, by substitution,

$$R = Q \left(\frac{A^2}{2} - 1 - \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} - \dots - \frac{1}{(2n+1)^2} \right).$$

But when $x = 1$, A is a quadrant, and $\frac{A^2}{2} = 1 + \frac{1}{3^2} +$

$\frac{1}{5^2} + \frac{1}{7^2} + \&c.$ continued indefinitely (see the solution to qu. 76);

therefore $\frac{A^2}{2} - 1 - \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} - \dots - \frac{1}{(2n+1)^2}$ must be

equal to the remaining terms $\frac{1}{(2n+3)^2}, \frac{1}{(2n+5)^2},$

$\frac{1}{(2n+7)^2}, \&c.$ and $\int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n+1} \dot{x}}{\sqrt{(1-x^2)}} =$

$$Q \left(\frac{1}{(2n+3)^2} + \frac{1}{(2n+5)^2} + \frac{1}{(2n+7)^2} + \frac{1}{(2n+9)^2} + \&c. \right).$$

SECOND SOLUTION, by Mr. CUNLIFFE.

Assume $\int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n+1} \dot{x}}{\sqrt{(1-x^2)}} = ax^{2n+3} + bx^{2n+5} + cx^{2n+7} + \&c.$

Taking the fluxions and dividing by $\frac{\dot{x}}{\sqrt{(1-x^2)}}$ gives $\int \frac{x^{2n+1} \dot{x}}{\sqrt{(1-x^2)}}$
 $=$

$$= \sqrt{1-x^2} \times \left\{ (2n+3)ax^{2n+2} + (2n+5)bx^{2n+4} + (2n+7)cx^{2n+6} + \&c. \right\}.$$

Again, taking the fluxions and dividing by $\frac{x^{2n+1} \dot{x}}{\sqrt{1-x^2}}$ we have

$$1 = (1-x^2) \cdot \left\{ (2n+2)(2n+3)a + (2n+4)(2n+5)bx^2 + (2n+6)(2n+7)cx^4 + \&c. \right\} \\ - (2n+3)ax^2 - (2n+5)bx^4 - (2n+7)cx^6 - (2n+9)dx^8 - \&c.$$

By actual multiplication, &c.

$$\left. \begin{aligned} (2n+2)(2n+3)a + (2n+4)(2n+5)bx^2 + (2n+6)(2n+7)cx^4 \&c. \\ - 1 - (2n+2)(2n+3)ax^2 - (2n+4)(2n+5)bx^4 \&c. \\ - (2n+3)ax^2 - (2n+5)bx^4 \&c. \end{aligned} \right\} = 0.$$

Making the homologous terms $= 0$, $a = \frac{1}{(2n+2)(2n+3)}$,

$$b = \frac{(2n+3)}{(2n+2)(2n+4)(2n+5)}, c = \frac{(2n+3)(2n+5)}{(2n+2)(2n+4)(2n+6)(2n+7)}, \&c.$$

$$\text{Again, } \int \frac{\dot{x}}{\sqrt{1-x^2}} \int \frac{\dot{x}}{\sqrt{1-x^2}} \int \frac{x^{2n+1} \dot{x}}{\sqrt{1-x^2}} = \int \frac{\dot{x}}{\sqrt{1-x^2}} \times \left\{ ax^{2n+3} + bx^{2n+5} \right. \\ \left. + cx^{2n+7} + \&c. \right\} = \int \frac{x^{2n+1} \dot{x}}{\sqrt{1-x^2}} \times \left\{ ax^2 + bx^4 + cx^6 + dx^8 \&c. \right\}$$

the fluent of which generated whilst x from 0 becomes 1, is Q

$$\times \left\{ \frac{(2n+2)a}{(2n+3)} + \frac{(2n+2)(2n+4)b}{(2n+3)(2n+5)} + \frac{(2n+2)(2n+4)(2n+6)c}{(2n+3)(2n+5)(2n+7)} + \&c. \right\}$$

where Q denotes the whole fluent of $\frac{x^{2n+1} \dot{x}}{\sqrt{1-x^2}}$. (See art. 286, vol. 2. Simpson's Fluxions.)

$$\text{Now } Q = \frac{2 \cdot 4 \cdot 6 \dots (n)}{1 \cdot 3 \cdot 5 \cdot 7 \dots (n)} = \frac{2 \cdot 4 \cdot 6 \dots (2n)}{3 \cdot 5 \cdot 7 \dots (2n+1)}, \text{ by art. 298}$$

of the same book. And by writing for a, b, c , &c. their values before determined, we get

$$Q \times \left\{ \frac{(2n+2)a}{(2n+3)} + \frac{(2n+2)(2n+4)b}{(2n+3)(2n+5)} + \frac{(2n+2)(2n+4)(2n+6)c}{(2n+3)(2n+5)(2n+7)} + \&c. \right\} \\ = Q \times \left\{ \frac{1}{(2n+3)^2} + \frac{1}{(2n+5)^2} + \frac{1}{(2n+7)^2} + \frac{1}{(2n+9)^2} + \&c. \right\}.$$

XXIX. QUESTION 239, by Mr. IVORY.

Let n be a whole positive number, and put

$$P = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots 2n} : \text{ then the whole value of the triple}$$

triple fluent $\int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{x^{2n} dx}{\sqrt{(1-x^2)}}$, which is generated while x increases from 0 to 1, is equal to

$$P \times \left\{ \frac{1}{(2n+2)^2} + \frac{1}{(2n+4)^2} + \frac{1}{(2n+6)^2} + \frac{1}{(2n+8)^2} + \&c. \right\} \times \frac{\pi}{2}$$

where π is the semi-circumference to the radius 1?

N.B. Observe that the case $n=0$ is included by taking $P=1$.

FIRST SOLUTION, by Mr. LOWRY.

Let A = the arch whose sine is x ; then

$$\int \frac{\dot{x}}{\sqrt{(1-x^2)}} = A, \int \frac{x^2 \dot{x}}{\sqrt{(1-x^2)}} = \frac{1}{2}A - \frac{1}{2}x \sqrt{(1-x^2)}$$

$$\int \frac{x^4 \dot{x}}{\sqrt{(1-x^2)}} = \frac{1.3}{2.4}A - \frac{1.3}{2.4}(x + \frac{2}{3}x^3) \sqrt{(1-x^2)}$$

&c. &c.

$$\text{and in general } \int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}} =$$

$$PA - P(x + \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 + \dots + \frac{2.4 \dots 2n-2}{1.3.5 \dots 2n-1}x^{2n-1}) \sqrt{(1-x^2)};$$

where the first term of each expression, when $A = \frac{\pi}{2}$, is the whole fluent generated while x increases from 0 to 1.

$$\begin{aligned} \text{Therefore } \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}} &= \\ P \int \left\{ A \dot{A} - (x\dot{x} + \frac{2}{3}x^3\dot{x} + \dots + \frac{2.4 \dots 2n-2}{1.3.5 \dots 2n-1}x^{2n-1}\dot{x}) \right\} & \\ = P \left(\frac{A^2}{2} - \frac{x^2}{2} - \frac{2}{3.4}x^4 - \dots - \frac{2.4 \dots 2n-2}{3.5.7 \dots 2n}x^{2n} \right). & \end{aligned}$$

And if R be put for the triple fluent

$$\int \frac{x}{\sqrt{(1-x^2)}} \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}}, \text{ we have } R =$$

$$P \int \frac{A^2 \dot{A}}{2} + P \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \left(-\frac{x^2}{2} - \frac{2}{3.4}x^4 - \dots - \frac{2.4 \dots 2n-2}{3.5.7 \dots 2n}x^{2n} \right).$$

$$\text{But the fluents } \int \frac{A^2 \dot{A}}{2}, \int \frac{x^2 \dot{x}}{\sqrt{(1-x^2)}}, \int \frac{x^4 \dot{x}}{\sqrt{(1-x^2)}} \text{ to}$$

$\int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}}$, generated while x increases from 0 to 1, being
 $\frac{\pi^2}{24} \cdot \frac{\pi}{2}, \frac{1 \cdot \pi}{2 \cdot 2}, \frac{1 \cdot 3 \cdot \pi}{2 \cdot 4 \cdot 2}, \frac{1 \cdot 3 \cdot 5 \cdot \pi}{2 \cdot 4 \cdot 6 \cdot 2}$ to $\frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \cdot \frac{\pi}{2}$,
 respectively; we have by substitution,

$$R = P \left(\frac{\pi^2}{24} - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \dots - \frac{1}{(2n)^2} \right) \cdot \frac{\pi}{2}.$$

But $\frac{\pi^2}{24} - \frac{\pi^2}{6} - \frac{\pi^2}{8} = (1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.) - (1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \&c.$ continued indefinitely (see solution to qu. 179).

Therefore $\frac{\pi^2}{24} - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \dots - \frac{1}{(2n)^2}$ is equal to the remaining terms $\frac{1}{(2n+2)^2}, \frac{1}{(2n+4)^2}, \frac{1}{(2n+6)^2} + \&c.$

$$\text{And } \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}} =$$

$$P \left(\frac{1}{(2n+2)^2} + \frac{1}{(2n+4)^2} + \frac{1}{(2n+6)^2} + \&c. \right) \times \frac{\pi}{2}.$$

SECOND SOLUTION, by Mr. CUNLIFFE.

By assuming $\int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}} = ax^{2n+2} + bx^{2n+4} + cx^{2n+6}$

+ &c. and proceeding as in qu. 238, we shall get $a = \frac{1}{(2n+1)(2n+2)}$,

$$b = \frac{(2n+2)}{(2n+1)(2n+3)(2n+4)}, c = \frac{(2n+2)(2n+4)}{(2n+1)(2n+3)(2n+5)(2n+6)}, \&c.$$

and therefore

$$\int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}} = \int \frac{\dot{x}}{\sqrt{(1-x^2)}} \times \{ ax^{2n+2} + bx^{2n+4} + cx^{2n+6} + \&c. \} = \int \frac{x^{2n} \dot{x}}{\sqrt{(1-x^2)}} \times \{ ax^2 + bx^4 + cx^6 + \&c. \}.$$

Now by art. 286, vol. 2, Simpson's Fluxions,

the fluent of the last expression generated while x from 0 becomes 1, is $Pq \times$

$$\left\{ \frac{(2n+1)a}{(2n+2)} + \frac{(2n+1)(2n+3)b}{(2n+2)(2n+4)} + \frac{(2n+1)(2n+3)(2n+5)c}{(2n+2)(2n+4)(2n+6)} + \&c. \right\}$$

where Pq denotes the whole fluent of $\frac{x^{2n}}{\sqrt{1-x^2}}$. And by art.

$$298, \text{ of the same book, } Pq = q \times \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots (2n)} = q \times \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)},$$

where q denotes the quadrantal arc of a circle, radius 1.

Again, by writing the above values of $a, b, c, \&c.$ Pq

$$\begin{aligned} & \times \left\{ \frac{(2n+1)a}{(2n+2)} + \frac{(2n+1)(2n+3)b}{(2n+2)(2n+4)} + \frac{(2n+1)(2n+3)(2n+5)c}{(2n+2)(2n+4)(2n+6)} + \&c. \right\} \\ & = \frac{Pq}{4} \times \left\{ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \frac{1}{(n+4)^2} + \&c. \right\}. \end{aligned}$$

XXX. PRIZE QUESTION 240, by Mr. IVORY.

Let a plane be drawn through the axis of an oblique cone, so as to be perpendicular to the plane of the base; and let a right-line be drawn to bisect the vertical angle of the triangle which is the common section of the plane and the cone: then, if a second plane be drawn through this line (*viz.* the line which bisects the vertical angle of the triangle) to cut the cone, the difference of two portions of the conic surface between the two planes, and comprehended in opposite angles, may be determined by means of a circular area and a rectilineal space?

SOLUTION, by Mr. IVORY, the Proposer.

Let CP (fig. 30. pl. D.) the height of the cone $= k$: the distance of the foot of CP from the centre of the base, OP, $= k$: the radius of the base $= r$: the distance of the point E, through which the plane CDF passes, $= r \times f$: the measure of the indefinite arc AD, $= \phi$: and the line CD, in the conic surface, $= s$. Then, $CD^2 = CP^2 + PR^2 + RD^2$; that is, in symbols,

$$\begin{aligned} s^2 &= k^2 + (k + r \cos \phi)^2 + r^2 \sin^2 \phi, \\ \text{or } s^2 &= k^2 + k^2 + r^2 + 2rk \cos \phi: \end{aligned}$$

The

The angle AED being denoted by ψ , then

$$\sin(\varphi - \psi) = f \sin \psi$$

$$\cos(\varphi - \psi) = \sqrt{1 - f^2 \sin^2 \psi}$$

$$\text{and } \cos \varphi = \cos(\varphi - \psi) \cos \psi - \sin(\varphi - \psi) \sin \psi$$

$$= \cos \psi \sqrt{1 - f^2 \sin^2 \psi} - f \sin^2 \psi.$$

Again, let $AC = a$, and $BC = b$, then

$$a^2 = h^2 + (k + r)^2$$

$$b^2 = h^2 + (k - r)^2$$

$$\text{consequently } \frac{a^2 + b^2}{2} = h^2 + k^2 + r^2 = \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2$$

$$\frac{a^2 - b^2}{2} = 2rk = 2\left(\frac{a+b}{2}\right) \cdot \left(\frac{a-b}{2}\right).$$

Therefore by substitution,

$$\begin{aligned} \epsilon^2 = & \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 - 2\left(\frac{a+b}{2}\right) \left(\frac{a-b}{2}\right) f \sin^2 \psi \\ & + 2\left(\frac{a+b}{2}\right) \left(\frac{a-b}{2}\right) \cos \psi \sqrt{1 - f^2 \sin^2 \psi} \end{aligned}$$

$$\text{or, } \epsilon^2 = \left(\frac{a+b}{2}\right)^2 \times \left\{ 1 + \left(\frac{a-b}{a+b}\right)^2 - 2\left(\frac{a-b}{a+b}\right) f \sin^2 \psi \right. \\ \left. + 2\left(\frac{a-b}{a+b}\right) \cos \psi \sqrt{1 - f^2 \sin^2 \psi} \right\}$$

but, according to the enunciation of the proposition, the point E is taken so that $f = \frac{a-b}{a+b}$: therefore

$$\epsilon^2 = \left(\frac{a+b}{2}\right)^2 \times \left\{ 1 + f^2 - 2f^2 \sin^2 \psi + 2f \cos \psi \sqrt{1 - f^2 \sin^2 \psi} \right\}$$

and, by extracting the square root of both sides,

$$\epsilon = \frac{a+b}{2} \times \left\{ f \cos \psi + \sqrt{1 - f^2 \sin^2 \psi} \right\}.$$

Because $\sin(\varphi - \psi) = f \sin \psi$

$$\cos(\varphi - \psi) = \sqrt{1 - f^2 \sin^2 \psi}$$

$$\text{we have } d\varphi - d\psi = \frac{d \sin(\varphi - \psi)}{\cos(\varphi - \psi)} = \frac{f \cos \psi \cdot d\psi}{\sqrt{1 - f^2 \sin^2 \psi}};$$

$$\text{therefore } d\varphi = \frac{d\psi}{\sqrt{1 - f^2 \sin^2 \psi}} \times \left\{ f \cos \psi + \sqrt{1 - f^2 \sin^2 \psi} \right\};$$

$$\text{also (because } \frac{a-b}{2} = \frac{a+b}{2} \times f)$$

$$d\epsilon = \frac{a-b}{2} \times \frac{-d\psi \sin \psi}{\sqrt{1 - f^2 \sin^2 \psi}} \times \left\{ f \cos \psi + \sqrt{1 - f^2 \sin^2 \psi} \right\},$$

$$\text{therefore by substitution, } \epsilon \sqrt{(r^2 d\varphi^2 - d\epsilon^2)} =$$

K 2

(a -

$$\frac{(a + b)r}{2} \times \frac{d\psi \sqrt{1 - \left(\frac{a-b}{2r}\right)^2 \sin^2 \psi}}{\sqrt{1 - f^2 \sin^2 \psi}} \times \{f \cos \psi + \sqrt{1 - f^2 \sin^2 \psi}\}.$$

Now $\oint \sqrt{r^2 d\psi^2 - d\epsilon^2}$ is double of the fluxion of the part of the conic surface bounded by the lines AC and CD*, which space is therefore (by putting $\epsilon = \frac{a-b}{2r}$) =

$$\left(\frac{a+b}{4}\right)r \times \int \frac{d\psi \sqrt{1 - \epsilon^2 \sin^2 \psi}}{\sqrt{1 - f^2 \sin^2 \psi}} \times \{f \cos \psi + \sqrt{1 - f^2 \sin^2 \psi}\}^2.$$

In the very same manner it may be shewn that the part of the conic surface bounded by the lines CB and CF is equal to

$$\left(\frac{a+b}{4}\right)r \times \int \frac{d\psi \sqrt{1 - \epsilon^2 \sin^2 \psi}}{\sqrt{1 - f^2 \sin^2 \psi}} \times \{-f \cos \psi + \sqrt{1 - f^2 \sin^2 \psi}\}^2.$$

Therefore the difference of the two said spaces is equal

$$(a+b)fr \times \int d\psi \cos \psi \sqrt{1 - \epsilon^2 \sin^2 \psi}$$

$$\text{or to } (a+b)r \times \frac{f}{\epsilon} \times \int \epsilon d\psi \cos \psi \sqrt{1 - \epsilon^2 \sin^2 \psi}$$

$$\text{or, by making } x = \epsilon \sin \psi, \text{ to } 2r^2 \times \int dx \sqrt{1 - x^2}.$$

Therefore by integrating, the difference of the two parts of the conic surface, that stand upon the arcs AD and BF, is equal to

$$r^2 \times \{x \sqrt{1 - x^2} + \text{Arc}(\sin = x)\}.$$

$$\text{Where } x = \frac{a-b}{2r} \times \sin \psi.$$

This formula suggests the following construction :

Describe the circle *adbe* (fig. 31. pl. D.), equal to the base of the cone ; in a diameter *ab*, take *of*, from the centre *o*, = $\frac{a-b}{2}$; draw *ed* through *f* so as to make the angle *ofd* = ψ ; and, lastly, draw the diameter *el*: then will the mixtilineal space *led* be equal to the difference of the two conic spaces between the planes (fig. 30. pl. D.) AFBC and DFEC, and comprehended in opposite angles.

The difference of the parts of the conic surface that lie on the opposite sides of the plane CFD (fig. 30.) is double of the space *led* (fig. 31.). For the difference of the conic spaces that stand on the arcs AF and BD is equal to the difference of the spaces that stand on the arcs AD and BF.

Mr. IVORY is requested to send to Mr. GLENDINNING's for the Medal for solving the Prize Question.

* See the second solution to question 16, page 39, vol. I.

NOTICES.

I. MATHEMATICAL WORKS LATELY PUBLISHED.

The Philosophical Transactions of the Royal Society of London, for 1808. The Mathematical Papers contained in this vol. are,

1. On a New Property of the Tangents of the three angles of a Plane Triangle. By Mr. Wm. Garrad, Quarter-Master of Instruction at the Royal Naval Asylum, at Greenwich.

2. On a New Property of the Tangents of three Arches trisecting the Circumference of a Circle. By Nevil Maskeline, D.D. F. R. S. and Astronomer Royal.

3. Observations of a Comet, made with a view to investigate its Magnitude and the nature of its Illumination. To which is added, an Account of a New Irregularity lately perceived in the apparent figure of the Planet Saturn. By Dr. Herschell.

4. Hydraulic Investigations, subservient to an intended Croonian Lecture on the Motion of the Blood. By Dr. Young.

The Transactions of the Royal Society of Edinburgh. Vol. VI. Part II. contains the following Mathematical Papers.

1. Of the Solids of Greatest Attraction, or those which, among all the Solids that have certain Properties, attract with the greatest Force in a given Direction. By Professor Playfair.

2. An Account of a very extraordinary Effect of Refraction, observed at Ramsgate, by the Rev. S. Vince.

3. New Series for the Quadrature of the Conic Sections, and the Computation of Logarithms. By Mr. Wm. Wallace.

An Essay on the various orders of Logarithmic Transcendentals. With an inquiry into their application to the Integral Calculus and the Summation of Series. By Wm. Spence. Quarto. p. p. 128. Price 12s.

II. MATHEMATICAL WORKS IN THE PRESS.

A New and Enlarged edition of Mr. Professor Playfair's *Proofs of the Huttonian Theory of the Earth* is in the Press, and will speedily be published in one volume, quarto.

Mr. Renoward, of Trinity College, Cambridge, has in the Press a *Treatise on Spherical Trigonometry*.

A *System of Conic Sections*, adapted to the Study of Natural Philosophy. By the Rev. D. M. Peacock, A. M. Formerly Fellow of Trinity College, Cambridge.

Tracts, Mathematical and Philosophical, consisting chiefly of original or unpublished Pieces. By Dr. Hutton. These Tracts will form about 2 or 3 Octavo Volumes, and may be considered as Dr. Hutton's select remaining Papers of Inventions, Discoveries, and Improvements in those Sciences.

OBITUARY.

III. Died at Fotheringham, near Dundee, on the 23rd of August, 1808, in the 75th year of his age, Robert Small, D.D., F. R. S. EDIN. One of the Ministers of Dundee. Dr. Small has been long known to the World not only as a learned Divine, but also as an ingenious Mathematician, and he has sufficiently shewn how well he was entitled to this last Character by the excellent Demonstrations he has given of some of Dr. Stewart's General Theorems in the 2nd vol. of the *Edin. Phil. Transactions*; and more recently by his Account of the Astronomical Discoveries of Kepler, published in the year 1804, in one vol. 8vo.

End of the First Part of the Second Volume,

W. Glendinning, Printer, Hatton Garden, London.

NEW SERIES
OF THE
MATHEMATICAL
REPOSITORY.

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VOL. II. PART II.
—•••••

ORIGINAL ESSAYS.
—•••••

ARTICLE I.

DEMONSTRATION of a Proposition in Mechanics. By A. B.

To the Editor of the Mathematical Repository.

SIR,

If you can insert the following Mechanical Proposition,
in your valuable Miscellany, without excluding matter of more
consequence, you will oblige

Sir,

Your's, &c.

A. B.

9th. May, 1806.

VOL. II. PART II.

PROPOSITION.

PROPOSITION.

If a system of bodies be connected together and supported at any point which is not the centre of gravity, and then left to descend by that part of their weight which is not supported; $4l$ multiplied into the sum of all the products of each body into the space it has perpendicularly descended, will be equal to the sum of all the products of each body into the square of its velocity; l being equal to $16\frac{1}{2}$ feet, or the space through which a heavy body descends in one second of time.

Let A, B, C, &c. (fig. 1. pl. A.) be a system of bodies connected together and supported at S, and let G, O, be the centres of gravity and oscillation; join SA, SB, SC, &c. Let SD be parallel and SL perpendicular to the horizon, and at the distance SO describe the quadrant DOL, and put s and c for the sine and cosine of the angle OSL to the rad. (1). In any small time t' , the velocity generated in a body at O, descending down the arch DL, $= 2t'ls$, and by the property of the centre of oscillation, the velocity generated in the same time in O, is likewise $= 2t'l \times s$, and therefore the velocity generated in

the centre of gravity G, will $= 2t'ls \times \frac{SG}{SO}$; but by mecha-

$$\text{nics } SO = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{(A + B + C + \&c.) \times SG};$$

therefore by substitution, the velocity generated in the centre of gravity in the time t' , $= 2t'ls$

$$\times \frac{SG^2 \times (A + B + C + \&c.)}{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}.$$

But because the velocities of A, B, C, &c. and G, are in the same proportion as their respective distances from S, and these distances are of the same dimension in both numerator and denominator; if, therefore, for the velocities of A, B, C, &c. and G, we write $v, u, w, \&c.$ and V , the equation becomes $2t'ls \times \frac{V^2 \times (A + B + C + \&c.)}{Av^2 + Bu^2 + Cw^2 + \&c.} = V$, or, $2t'ls \times V \times (A + B + C + \&c.) = Av^2 + Bu^2 + Cw^2 + \&c.$ Let $v', u', w', \&c.$ and V' , be the respective velocities of A, B, C, &c. and G, in a direction perpendicular to the horizon, then because $V : V' :: 1 : s$, $V = \frac{V'}{s}$, hence, $2t'lV' \times (A + B + C + \&c.)$

$$= Av'^2 + Bu'^2 + Cw'^2 + \&c.$$

$= Av^2 + Bu^2 + Cw^2 + \&c. = 2t'l \times (-Av' + Bu' + Cw' + \&c.)$ by the nature of the centre of gravity. Suppose $x, y, z, \&c.$ to represent the spaces passed over in a perpendicular direction in the time t' ; then because in equally accelerated motions, the spaces passed over are as the times multiplied into the last velocities, therefore $x'' \times 2t' : v' \times t' :: t' : x$, therefore $v' = \frac{2x}{t'}$, and in like manner $u' = \frac{2y}{t'}$, $w' = \frac{2z}{t'}$, &c. hence by substitution $4l \times (-Ax + By + Cz + \&c) = Av^2 + Bu^2 + Cw^2 + \&c.$ Now as neither t' nor s enter into the expression, the result is not confined to any particular time or position; it is therefore evident that the equation is general as expressed in the proposition; for it is well known to mathematicians, that if in the conclusion of a problem all those quantities vanish that restricted it to any particular case, then the result, or equation becomes *general* for those quantities which remain; wherefore the true relation between the spaces perpendicularly descended by the bodies and the squares of their velocities has been determined in all cases whatever.

COR. Because the velocity generated in the centre of gravity in a direction perpendicular to SO in the time $t' = 2t'ls \times \frac{SG}{SO}$, the velocity generated in the same time in a direction perpendicular to the horizon must be equal $2t'ls^2 \times \frac{SG}{SO}$, and $2t'l =$ velocity that would be generated in the same time by the whole weight of the system (W); therefore $2t'l : 2t'ls^2 \times \frac{SG}{SO} :: W : s^2W \times \frac{SG}{SO} =$ that part of the weight which is not supported, or which generates the perpendicular velocity in the centre of gravity. Hence it is evident, that the pressure upon S at that instant is equal to the remaining part of the whole body $= W \times (1 - s^2 \times \frac{SG}{SO})$.

Ex. 1. From Simpson's Traacts, page 131.

Suppose that a thread $ACnCA$ (fig. 2. pl. A.), having two equal weights A, A , suspended at the ends thereof, is hung over two tacks C, C , in the same horizontal line; and that to the middle point of the thread (n) equally distant from the tacks, another given weight B is fixed, which is permitted to descend by

its own gravity, so as to cause the other two weights to ascend : it is proposed to find the law of the velocity by which the said weights ascend and descend, &c. &c.

Let bn be an indefinitely small space through which the body has descended, join Cb and draw br perpendicular to Cn ; then the velocity of A : velocity of B :: rn : bn :: En : Cn by similar triangles. If therefore $CE = a$, $Cn = y$, $En = x$, and

$u =$ velocity of B , then the velocity of $A = \frac{ux}{y}$. The space

ascended by A , $A = y - a$ and the space descended by $B = x$; therefore, by substituting in the general theorem we shall have

$$4l \times (2Aa - 2Ay + Bx) = 2A \times \frac{u^2 x^2}{y^2} + Bu^2;$$

hence, the equation reduced gives

$$u^2 = 4ly^2 \times \frac{Bx - 2Ay + 2Aa}{By^2 + 2Ax^2}, \text{ and therefore}$$

$$u = 2y \sqrt{\left(l \times \frac{Bx - 2Ay + 2Aa}{By^2 + 2Ax^2} \right)} = \text{velocity of } B.$$

$$\text{The velocity of } A, A = \frac{ux}{y} = 2x \sqrt{\left(l \times \frac{Bx - 2Ay + 2Aa}{By^2 + 2Ax^2} \right)}.$$

Simpson's expression for the velocities of B and A , when properly reduced, perfectly agree with those given above.

Ex. 2. Let the weights A and D (fig. 3. pl. A) be unequal, and suppose the weight B to descend in the straight line EB , E being the centre of gravity of A and D when placed at C and C , and let the several lines be expressed as is represented in the figure; then by substitution, in the general theorem given above, we have

$$4l \times Bx - A \times (y - a) - D \times (z - b) = A \times \frac{x^2 u^2}{y^2} + D \times \frac{x^2 u^2}{z^2}.$$

$$\text{Reduced, } u^2 = 4ly^2 z^2 \times \frac{Bx - A \times (y - a) - D \times (z - b)}{Ax^2 z^2 + By^2 z^2 + Dx^2 y^2}.$$

If x be given, y and z are known; E being the centre of gravity of A and D , $Aa = Db$.

If in the value of u^2 just found, we suppose $a = b$, $z = y$, and $A = D$, u^2 will be the same as in the last example.

Ex. 3. Let a cylinder of heavy metal roll down a plane which makes an angle with the horizon whose sine and cosine $= s$ and c , and suppose the friction just sufficient to keep it from sliding; it is required to find the angle which another plane makes with the horizon, so that the same cylinder moving

down it, the spaces passed over by the sliding and rolling motions in the same time, may be to each other as $n : 1$; the friction varying as the pressure against the plane.

Let S and C = sine and cosine of the second plane, w = weight of the cylinder, $l = 16\frac{1}{2}$ feet, t = time from the beginning of the motion, x = space passed over by the rolling motion, and v = velocity generated in the circumference of the cylinder round its axis in the same time; then it is well known

that $\frac{v}{\sqrt{2}}$ = corresponding velocity of the centre of gyration of

the cylinder. If the same body be placed upon different inclined planes, the forces down these planes vary as the sines of the angles which they make with the horizon and the pressure against them $\propto$ the cosines of the same angles. Hence, Sw = force down the first plane, and the force of friction

= $\frac{ws}{3}$ (see Emerson's Mechanics, Prop. 61, Schol.); therefore

the friction upon the second plane = $\frac{Cws}{3c}$; but the force down

this plane = Sw , therefore the remaining force when that of

friction is subtracted = $Sw - \frac{Cws}{3c} = w \times \frac{3cS - Cs}{3c}$; hence

this proportion, as $w \times 1^n : w \times \frac{3cS - Cs}{3c} \times t^2 :: l : l^n$

$\times \frac{3cS - Cs}{3c}$ = whole space moved down the plane by both the sliding and rolling motions.

The force $\frac{Cws}{3c}$ is entirely employed in generating the rotatory motion of the cylinder, and may be considered as a weight without inertia, descending through the space x ; therefore by the

theorem $4l \times \frac{Cws}{3c} \times x = w \times \frac{v^2}{2} = 2w \times \frac{2x^2}{t^2}$, for by me-

chanics $v = \frac{2x}{t}$, hence $l^2 \times \frac{2Cs}{3c} = x$ = space passed over by

rolling; therefore space passed over by sliding = $l^2 \times$

$\left(\frac{2cS - Cs}{3c} - \frac{2Cs}{3c}\right) = l^2 \times \frac{3cS - 3Cs}{3c}$. Hence space

passed over by sliding : space passed over by rolling :: $3cS - 3Cs : 2Cs :: n : 1$, by the hypothesis; therefore $3(cS - Cs) = 2nCs$, or $3cS = (2n + 3)Cs$, and $9c^2S^2 =$

$$(2n+3)^2 \times C^2 s^2 = (2n+3)^2 \times (1-S^2) \times s^2; \text{ reduced } S^2 = \frac{(2n+3)^2 \times s^2}{9c^2 + (2n+3)^2 \times s^2}, \text{ therefore } S = \frac{(2n+3)s}{\sqrt{9c^2 + (2n+3)^2 \times s^2}}.$$

COR. If t be given, the space passed over by both or either of these motions will be known; and if the length of the plane, or the space passed over be given, the time t may be found.

The same theorem much facilitates the solution of a great number of difficult mechanical problems, and applies with much advantage to a very considerable part of the propositions in Sect. 6. Atwood on Rectilinear and Rotatory Motion. For example, take Prop. 13.

Put x = distance of the centre of gyration from the centre of gravity G , s = space descended by P in the time t , and v = velocity generated in P , the other symbols as in that Prop.

Then $d : x :: v : \frac{vx}{d}$ = velocity of the centre of gyration;
 hence, by the theorem, $4l \times Ps = P \times v^2 + w \times \frac{v^2 x^2}{d^2}$
 $= \frac{Pd^2 + wx^2}{d^2} \times v.$

But by mechanics $v = \frac{2s}{t}$, therefore by substitution and reduction, it will appear that $x^2 = \frac{lt^2 d^2 P - sd^2 P}{ws} = \frac{Pd^2}{w} \times \frac{lt^2 - s}{s}$,
 therefore $x = d \sqrt{\left(\frac{P}{w} \times \frac{lt^2 - s}{s}\right)}$ = distance of the centre of gyration required. Because $4lP \times \frac{tv}{2} = \frac{Pd^2 + wx^2}{d^2} \times v^2$,
 therefore $v = 2l \times \frac{Ptd^2}{Pd^2 + wx^2} = 2l \times \frac{Pd^2}{Pd^2 + wx^2}$ when $t=1$,
 and is therefore = the accelerating force when the force of gravity = $2l$. But $x^2 = bd$, therefore the accelerating force of P ,
 or the point $D = v = 2l \times \frac{Pd^2}{Pd^2 + bdw} = 2l \times \frac{Pd}{Pd + bw}.$



ARTICLE II.

On the Motion of Pendulums whose points of suspension are moveable. In a Letter from Mr. John Gough.

To the Editor of the Mathematical Repository.

Middleshaw, near Kendal.
12th. May 1805.

SIR,

The Author of the following Propositions, is not ignorant that they contradict a remark made on the answer to the 648th question of the Gentleman's Diary, by a late eminent mathematician. The knowledge of this circumstance does not however forbid him to hope that his essay will meet with a candid reception on the part of your readers, should it be thought worthy a place in the Mathematical Repository; because he has endeavoured to oppose a careful investigation of the subject to bare assertion; which is of no value when unsupported by demonstrative evidence, for the greatest men are subject to error as oft as they neglect rigid demonstration, which is the only true road to geometrical certainty.—The elementary work quoted in the present paper, is a Treatise of Mechanics lately published, by Mr. Olinthus Gregory of the Royal Military Academy at Woolwich, which cannot fail of exciting attention, not only by its copiousness and the quantity of original matter, but also by the many valuable and judicious selections the author has made from the foreign mathematicians.

I am, &c.

JOHN GOUGH.

PROPOSITION I.

IF a given pendulum vibrate upon a pin, fixed in the centre of gravity of a given vessel, which is supported by a perfectly smooth and horizontal plane; the common centre of gravity of the pendulum and the vessel will descend and ascend alternately, in a given right line perpendicular to the horizon; while
the

the centre of gravity of the pendulum describes a portion of an ellipse, having for its semi-transverse the given vertical line properly produced, and for its semi-conjugate the distance of the same centre of gravity from that of the system.

DEMONSTRATION. Let the vertical plane AC (fig. 4. pl. A.) represent the vessel, or more properly, that section of it, in which the pendulum PG vibrates; P the centre of gravity of AC; also let PG be a position of the pendulum, G its centre of gravity, and T the common centre of the system AC PG; through P draw FE parallel to the horizon, and through T, NM perpendicular to FE. Now the external forces which give motion to the system ACPG are, 1st. the weight of the pendulum which acts upon its centre of gravity G (Mechanics book 1, art. 106); 2^d. the reaction of the plane CD, upon the point P, which is parallel to the former force, but in a contrary direction; consequently the resultant of these two forces is equal to their difference (ibid. 73); and by the nature of the centre of gravity it acts upon the system ACPG at the point T, and in the direction NM. But all the other forces, acting upon T, arise from the mutual action and reaction of AC and PG; therefore the latter forces do not disturb the motion of T, their common centre of gravity; hence it follows, from the doctrine of forces, that T will always be found in the right line NM, the position of which is thus found. Let p be the place of the centre of gravity of AC, when PG is in an horizontal position, coinciding with FE; make $pN = PT$, which is given in magnitude; therefore the point N is given, and NM is perpendicular to FE. Make $NM = PT$, NK and $pR = PG$; then $NR = MK = TG$; and when P coincides with p , G will be at R; also when P has moved through $pN = PT$, G will be at K; hence RCK is a quadrant of an ellipse, having NK for its semi-transverse and NR for its semi-conjugate (Euc. Con. Ellip. prop. 30). Q. E. D.

COR. 1. Let $b =$ the weight of the pendulum, $w' =$ that of the vessel; $l = PG$, the distance between the point of suspension and the pendulum's centre of gravity; and we have $\frac{bl}{b + w'} = PT = pN$; hence if $w' = 0$ (fig. 4, pl. A.). $pN = PR$ and G descends in a vertical line passing through R while P moves through the space pR .

COR. 2. But if w' be infinite, pN or its equal $\frac{bl}{b + w'} = 0$, in which case N coincides with p , $NR = NK$; that is, the ellipse becomes a circle and the point of suspension is fixed at p .

PROPOSITION

PROPOSITION II.

LET PG be a pendulum (fig. 4. pl. A.) whose point of suspension is P, and let it be urged perpendicular to PG, by a force k acting at G, its centre of gravity; then put b = matter in PG, r = radius of gyration of PG to the centre G, q = its radius of gyration to the centre P; and the reaction of the pin P, perpendicular to the line PG, will be $= \frac{r^2 k}{q^2 + r^2}$.

DEMONSTRATION. For the pendulum, when in motion, must revolve about the points G and P with equal angular velocities, consequently the force k is divided into two less forces, one acting at G and the other at P perpendicular to PG (Mech. art. 74.): put the latter $= x$, and the former $= k - x$; then the angular velocity, generated in a given time and about the

centre G by the force x , is in a constant proportion to $\frac{GP \times x}{br^2}$

(ibid. art. 301, cor. 2, and art. 310.); for the same reason, the angular velocity generated about P in the same time by $k - x$

has an equal ratio to $\frac{PG \times (k - x)}{bq^2}$: but the two angular

velocities are equal; hence $\frac{x}{r^2} = \frac{k - x}{q^2}$; consequently $x =$

$$\frac{r^2 k}{q^2 + r^2} \quad Q. E. D.$$

CON. Draw GS perpendicular to FE; also put $PG = l$, $PS = n$, $r^2 + q^2 = c^2$, and that part of b which acts perpen-

dicular to PG $= \frac{nb}{l} = k$; hence $x = \frac{r^2 nb}{lc^2}$ therefore $k - x =$

$\frac{q^2 nb}{lc^2}$: hence that part of $k - x$, which acts parallel to SP, in

the direction NT $= \frac{q^2 n^2 b}{l^2 c^2} = f$, or the motive force of the

system ACPG. Consequently if there be two pendulums PG, QI, which are every way equal, and make equal angles with the horizontal lines FE, fe (fig. 4. pl. A.), the motive forces acting upon T, t, the centres of gravity of the systems ACPG, acQI, will be equal; because the quantities b, l, n, q, r , are equal in both cases.

PROPOSITION III.

If two bodies A, C, (fig. 5. pl. A.) be impelled in the right lines AB, CD, by motive forces, which are constantly equal to each other, though each force may be variable in respect of itself; the spaces described by A, C, in equal times, will always be inverfely as their quantities of matter or weights, that is, as $\frac{1}{A}$ to $\frac{1}{C}$.

DEMONSTRATION. Put y, z = these spaces; t, v , the corresponding velocities of A, C; then fince the motive force of A is always equal to that of C, the motion generated in A will always be equal to that generated in C in the same time. But the velocity is inverfely as the matter when the motion is given: hence $t : v :: \frac{1}{A} : \frac{1}{C}$; but as $t : v :: \dot{y} : \dot{z}$; therefore $\dot{y} : \dot{z} :: \frac{1}{A} : \frac{1}{C}$; consequently y is to z as $\frac{1}{A}$ is to $\frac{1}{C}$ (Emerson's Fluxions, prop. 2, sect. 1.) Q. E. D.

COR. If two spaces, AB, CD, run over by two bodies, A, C, impelled as above, be inverfely as their quantities of matter; these spaces are described in equal times. For if CD be not described in the same time with AB, let some other line as CE be the contemporary space, which is greater or less than CD by an assignable magnitude EC. Then we have by the proposition, as $AB : CE :: \frac{1}{A} : \frac{1}{C}$, and by hypothesis, as $AB : CD :: \frac{1}{A} : \frac{1}{C}$; hence $CE = CD$ which is absurd; consequently AB, CD, are described in equal times.

PROPOSITION IV.

LET PG, QI, (fig. 4. pl. A.) be two equal and similar pendulums, which are suspended as in prop. 1, at P, Q, the centres of gravity of the vessels AC, ac; also let them begin to descend at the same moment from the situations PG, QI, in which they are equally inclined to the horizon, and parallel to each other: they will arrive at the vertical positions NM, nm in

in the same time, whatever be the weights of the two vessels; that is, their times of vibrating through equal angles are equal.

DEMONSTRATION. Let T, t , be the centres of gravity of the systems $ACPG, acQI$; put the common weight of the pendulums $= b$; weight of $AC = w$; that of $ac = W$; the lines $PG, QI = l$; $NT = S$; $nt = s$: now the points T, t , will always be in the right lines MN, mn , by prop. 1; also NM

$= PT$, and $nm = Qt$, ibid.; hence PT or $NM = \frac{bl}{b+w}$;

and Qt or $nm = \frac{bl}{b+W}$ (cor. 1, prop. 1.); therefore, as PT

: $Qt :: \frac{1}{b+w} : \frac{1}{b+W}$; that is, PT is to Qt inversely as the

quantities of matter $ACPG, acQI$, which are to be moved through the right lines TM, tm . Moreover the angles TPN, tQn , are equal by hypothesis at the commencement of motion; therefore the motive forces acting upon T, t , in the lines $NT,$

nt , are equal (cor. prop. 2.): hence as $\dot{S} : \dot{s} :: \frac{1}{b+w} : \frac{1}{b+W}$

(prop. 3.): but the triangles NPT, nQt , are similar, being right-

angled at N, n ; hence also, as $PT : Qt :: NT : nt :: \frac{1}{b+w} :$

$\frac{1}{b+W}$; consequently as $PT : Qt :: S + \dot{S} : s + \dot{s}$; that is,

the motive forces acting upon T, t , will continue equal between themselves, because the variable triangles, NPT, nQt , will always be similar; but $NM = PT$, and $nm = Qt$; therefore

as $NM : nm :: NT : nt$; hence, as $TM : tm :: \frac{1}{b+w} :$

$\frac{1}{b+W}$. Now when PG, QI become vertical, the points T, t , co-

incide with M, m , after moving through the spaces TM, tm ; these spaces are therefore described in the same time (cor. prop. 3.); that is, the pendulums vibrate through equal angles in equal times. Q. E. D.

COR. If W be infinite, I the centre of gravity of QI vibrates about the fixed point Q (cor. 2. prop. 1.); consequently the pendulum PG is synchronous to the pendulum QI , which being equal and similar to PG has its point of suspension fixed.

PROPOSITION V.

LET the vertical plane, CVD (fig. 6. pl. A.), represent a frustum of a sphere, which is supported by a horizontal plane touching it at V; let P be the centre of the sphere, G the centre of gravity of the frustum; also through P draw the horizontal line PR, and make GR perpendicular to RP, in which, take $RK = PG$: the point P will move over the space PR while the point G descends to K; and this will happen while a similar frustum of an equal sphere vibrates on its spherical centre, through an angle equal to VPG.

DEMONSTRATION. Complete the spherical surface AVD, which will be supported by the reaction of the touching plane at V, exerted along VP; therefore, AVD is a vessel without weight, upon whose centre of gravity P, the pendulum PG revolves; consequently, G will descend from G to K, while P moves through PR; that is, while the radius PG describes the angle GPV (cor. 1, prop. 1). Let now the weight of AVD become infinite, and G will describe the angle GPV in the same time (cor. prop. 1.). Q. E. D.

COR. Since the angular velocity of G about P is the same whether P be fixed or not: let v denote the velocity of G in the circle GL, when GR is its distance from PR, which velocity may be found from the fixed pendulum; then because GP, GR, are perpendiculars to a tangent at G and to PR, we have as $GP : GR :: v : \frac{v \times GR}{GP} =$ contemporary velocity of P in PR.

ARTICLE III.

An ocular demonstration of the forty-seventh proposition of the first book of Euclid. By Mr. H. DOUGLAS.

LET BEL be a right angled triangle, having the right angle LBE; the square DELM described upon the side LE, is equal to the square ABEF described upon the side BE, together with the square BIQL described upon the side BL. Fig. 7, pl. A.

Produce AB to C making BC equal to AB, draw CG parallel

parallel to BE meeting FE produced in G: then BCGE is a square and equal to the square ABEF. Join CI, GD, and produce DE to K; draw KH parallel to LE, and MN perpendicular to CG.

In the triangles BEL, KEF; the side BE is equal to the side EF, the angle LEB to the angle KEF, and the angles at B and F are equal because they are right angles, the third angles are therefore equal, and the triangles equal in all respects; KE, therefore, is equal to EL the side of the hypothenusal square, and KF to BL the side of the lesser square. The remaining space AKEB of the square AFEB is therefore equal to the space LEGC; and in the triangles, AHK, CLP, the sides AK, CL, and the angles AKH, CLP, being respectively equal, and the angles at A and C right angles, the triangles are equal in all respects; the space EBHK is therefore equal to the space GELP. Again, in the triangles LBE, GED, the side BE is equal to the side EG, the side LE to the side DE, and the angle BEL to the angle GED, the triangles are therefore identical and equal in all respects; the side GD is, therefore, equal to the side LB, and the angle EGD to the angle LBE, that is, the angle EGD is a right angle; therefore the points C, G, D, are in a straight line.

Again, the triangle MND is equal to the triangle LBE, because LE and MD are equal being sides of the same square, the angles at B and N are right angles, and the angle MDN equal to the angle BEL, the third angles are therefore equal and the triangles identical; MN then is equal to BL the side of the square BIOL, and the triangle MDN equal in all respects to the triangle EKF.

The triangles GDE, CBI, having the side GE equal to the side CB, the side GD to the side BI (or BL), and the angles at G and B right angles, are equal in all respects; the angle ICB therefore is equal to the angle GED, that is, equal to the angle PLC; therefore CI is parallel to LP, and LO to CP; wherefore the figure COLP is a parallelogram, and the triangle OLC equal to the triangle CLP.

Now the triangle CBI is made up of the trapezoid IBOL and the triangle COL or the triangle CLP; the triangle GED is therefore equal to the trapezoid IBOL and the triangle CLP; or if the triangle ERS be made equal to the triangle COL (which is equal to the triangle AKH), then the trapezoid IBOL is equal to the trapezoid GRSD.

Again, the triangles PNM, IOQ, having the side MN equal to the side IQ or BL, and the angles MPN, QOI, (each equal to the opposite angles CPL, COL, of the parallelogram COLP) are equal, and the angles at N and Q (right angles) also equal, the triangles are therefore identical or equal in all respects.

The square of the hypotenuse then contains the trapezoid GRSD, the trapezium LEGP and the triangles RES, MND and PNM which are together equal to the sum of the squares AFEB and IBLQ.

ARTICLE IV.

An investigation of some theorems which are of use in obtaining the sums of certain infinite series by means of circular arcs.

By Mr. JAMES CUNLIFFE, R. M. College.

1. LET $\frac{1 - cx}{1 - 2cx + x^2}$ be expanded in an infinite series.

Put $\frac{1 - cx}{1 - 2cx + x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$

multiplying the expression by $1 - 2cx + x^2$ gives

$$\begin{aligned} 1 - cx &= 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c. \\ &\quad - 2cx - 2cAx^2 - 2cBx^3 - 2cCx^4 - \&c. \\ &\quad + x^2 + Ax^3 + Bx^4 + \&c. \end{aligned}$$

whence

$$\begin{aligned} 0 &= Ax + Bx^2 + Cx^3 + Dx^4 + \&c. \\ &\quad - cx - 2cAx^2 - 2cBx^3 - 2cCx^4 - \&c. \\ &\quad + x^2 + Ax^3 + Bx^4 + \&c. \end{aligned}$$

making the coefficients of the like powers of $x = 0$, we get

$$A = c; B = 2cA - 1; C = 2cB - A; D = 2cC - B; \&c.$$

Let c be the cosine of a circular arch z , radius 1; then by a known theorem $B = 2c^2 - 1 = \cos. 2z$;

$$C = 2cB - c = \cos. 3z; D = 2cC - B = \cos. 4z, \&c.$$

$$\begin{aligned} \text{Therefore } \frac{1 - cx}{1 - 2cx + x^2} &= 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c. \\ &= 1 + x. \cos. z + x^2. \cos. 2z + x^3. \cos. 3z + x^4. \cos. 4z + \&c. \end{aligned}$$

2. Take

2. Take $x = 1$, then $\frac{1 - cx}{1 - 2cx + x^2} = \frac{1 - c}{2(1 - c)} =$

$$\frac{1}{2} = 1 + \text{cof. } z + \text{cof. } 2z + \text{cof. } 3z + \text{cof. } 4z + \&c.$$

or, $-\frac{1}{2} = \text{cof. } z + \text{cof. } 2z + \text{cof. } 3z + \text{cof. } 4z + \&c.$

multiplying this expression by $\frac{z}{2}$ and taking the fluents

$$-\frac{z}{2} = \text{fin. } z + \frac{1}{2} \text{ fin. } 2z + \frac{1}{3} \text{ fin. } 3z + \frac{1}{4} \text{ fin. } 4z + \&c.$$

Put q = the quadrantal arc of a circle, radius 1, then when $z = 2q$, the sines of z , $2z$, $3z$, $4z$, &c. will be each = 0; therefore the correct equation of the fluents will be

$$q - \frac{z}{2} = \text{fin. } z + \frac{1}{2} \text{ fin. } 2z + \frac{1}{3} \text{ fin. } 3z + \frac{1}{4} \text{ fin. } 4z + \&c.$$

3. Take $x = -1$, then $\frac{1 - cx}{1 - 2cx + x^2} = \frac{1 + c}{2(1 + c)} =$

$$\frac{1}{2} = 1 - \text{cof. } z + \text{cof. } 2z - \text{cof. } 3z + \text{cof. } 4z - \&c.$$

whence by transposition

$$\frac{1}{2} = \text{cof. } z - \text{cof. } 2z + \text{cof. } 3z - \text{cof. } 4z + \&c.$$

multiplying this expression by $\frac{z}{2}$ and taking the fluents

$$\frac{z}{2} = \text{fin. } z - \frac{1}{2} \text{ fin. } 2z + \frac{1}{3} \text{ fin. } 3z - \frac{1}{4} \text{ fin. } 4z + \&c.$$

Taking half the sum of the preceding expression and that deduced at art. 2, and we shall have

$$\frac{q}{2} = \text{fin. } z + \frac{1}{3} \text{ fin. } 3z + \frac{1}{5} \text{ fin. } 5z + \&c.$$

Also taking their difference

$$q - z = \text{fin. } 2z + \frac{1}{2} \text{ fin. } 4z + \frac{1}{3} \text{ fin. } 6z + \&c.$$

The foregoing theorems are obtained in a different manner at the beginning of Landen's 5th. Memoir.

ARTICLE V.

An investigation of some theorems which are of use in obtaining the sums of certain infinite series by means of hyperbolic logarithms. By Mr. JAMES CUNLIFFE.

1. LET the expression $\frac{s}{1 - 2cx + x^2}$ be expanded in an infinite series.

$$\begin{aligned} \text{Put } \frac{s}{1 - 2cx + x^2} &= s + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \&c. \\ \text{and multiplying both sides of the expression by } 1 - 2cx + x^2, \\ s &= s + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \&c. \\ &\quad - 2csx - 2cAx^2 - 2cBx^3 - 2cCx^4 - 2cDx^5 - \&c. \\ &\quad + sx^2 + Ax^3 + Bx^4 + Cx^5 + \&c. \end{aligned}$$

making the coefficients of the homologous terms = 0;

$$A = 2sc; B = 2cA - s; C = 2cB - A; D = 2cC - B, \&c.$$

Now if s and c denote the sine and cosine of a circular arch z , radius 1: then by known theorems

$$\begin{aligned} A &= 2cs = \sin. 2z; B = 2cA - s = \sin. 3z; C = 2cB - A = \sin. 4z; \\ D &= 2cC - B = \sin. 5z, \&c. \end{aligned}$$

$$\begin{aligned} \text{and therefore } \frac{s}{1 - 2cx + x^2} &= s + Ax + Bx^2 + Cx^3 + Dx^4 + \&c. \\ &= \sin. z + x \cdot \sin. 2z + x^2 \cdot \sin. 3z + x^3 \cdot \sin. 4z + x^4 \cdot \sin. 5z + \&c. \end{aligned}$$

2. Take $x = 1$, then

$$\frac{s}{1 - 2cx + x^2} = \frac{s}{2(1-c)} = \sin. z + \sin. 2z + \sin. 3z + \sin. 4z + \&c.$$

$$\text{and it is well known that } \frac{\dot{c}}{s} = -\dot{z}$$

multiplying the former expression by this

$$\frac{\dot{c}}{2(1-c)} = -\dot{z} \times (\sin. z + \sin. 2z + \sin. 3z + \sin. 4z + \&c.)$$

taking the fluents

$$\frac{1}{2} \times \text{h.l.} \left(\frac{1}{1-c} \right) = \cos. z + \frac{1}{2} \cos. 2z + \frac{1}{3} \cos. 3z + \frac{1}{4} \cos. 4z + \&c.$$

But when $c = 0$, then the cosines of z , $3z$, $5z$, $\&c.$

will each = 0; and the cosines of $2z$, $4z$, $6z$, $8z$, $\&c.$

will be $-1, +1, -1, +1, \&c.$; therefore the correct equation

tion of the fluents will be

$$\begin{aligned} \frac{1}{2} \times \text{h. l.} \left(\frac{1}{1-c} \right) &= \left\{ \text{cof. } z + \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z + \frac{1}{4} \text{cof. } 4z + \&c. \right. \\ &\quad \left. + \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \&c. \right. \\ &= \left\{ \text{cof. } z + \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z + \frac{1}{4} \text{cof. } 4z + \&c. \right. \\ &\quad \left. + \frac{1}{2} \times \text{h. l. } 2; \right. \end{aligned}$$

whence, by transposition,

$$\frac{1}{2} \times \text{h. l.} \left(\frac{1}{1-c} \right) - \frac{1}{2} \times \text{h. l. } 2 = \frac{1}{2} \times \text{h. l.} \frac{1}{2(1-c)} = \text{cof. } z + \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z + \frac{1}{4} \text{cof. } 4z + \&c.$$

3. Again, take $x = -1$, then,

$$\frac{s}{1-2cx+x^2} = \frac{s}{2(1+c)} = \text{fin. } z - \text{fin. } 2z + \text{fin. } 3z - \text{fin. } 4z + \&c.$$

multiplying this expression by $\frac{c}{s} = -z$, gives

$$\frac{c}{2(1+c)} = -z \times (\text{fin. } z - \text{fin. } 2z + \text{fin. } 3z - \text{fin. } 4z + \&c.)$$

and taking the fluents

$\frac{1}{2} \times \text{h. l.} (1+c) = \text{cof. } z - \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z - \frac{1}{4} \text{cof. } 4z + \&c.$
But when $c=0$, the cofines of $z, 3z, 5z, \&c.$ will be each $= 0$;
and the cofines of $2z, 4z, 6z, 8z, \&c.$ will be $-1, +1, -1, +1, \&c.$; wherefore the correct equation of the fluents will be

$$\begin{aligned} \frac{1}{2} \times \text{h. l.} (1+c) &= \left\{ \text{cof. } z - \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z - \frac{1}{4} \text{cof. } 4z + \&c. \right. \\ &\quad \left. - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \&c. \right. \\ &= \left\{ \text{cof. } z - \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z - \frac{1}{4} \text{cof. } 4z + \&c. \right. \\ &\quad \left. - \frac{1}{2} \times \text{h. l. } 2; \right. \end{aligned}$$

therefore, by transposition,

$$\frac{1}{2} \times \text{h. l.} (1+c) + \frac{1}{2} \times \text{h. l. } 2 = \frac{1}{2} \times \text{h. l.} \{ 2(1+c) \} = \text{cof. } z - \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z - \frac{1}{4} \text{cof. } 4z + \&c.$$

And from what has been deduced before

$$\frac{1}{2} \times \text{h. l.} \frac{1}{2(1-c)} = \text{cof. } z + \frac{1}{2} \text{cof. } 2z + \frac{1}{3} \text{cof. } 3z + \frac{1}{4} \text{cof. } 4z + \&c.$$

half the sum of these two expressions, by the nature of logarithms, is

$$\frac{1}{4} \times \text{h. l.} \left(\frac{1+c}{1-c} \right) = \text{cof. } z + \frac{1}{2} \text{cof. } 3z + \frac{1}{3} \text{cof. } 5z + \&c.:$$

also, taking their difference,

$$\frac{1}{4} \times \text{h. l.} \frac{1}{4(1-c^2)} = \text{h. l.} \frac{1}{2\sqrt{1-c^2}} = \text{cof. } 2z + \frac{1}{2} \text{cof. } 4z + \frac{1}{3} \text{cof. } 6z + \&c.$$

The above theorems are obtained by a method materially different in Mr. Landen's 5th. Memoir, Art. 16.

ARTICLE VI.

Solution of a Dynamical Question. By Mr. JOHN BARRY.

To the Editor of the Mathematical Repository.

SIR,

In Art. 6, part 2, of the first vol. of the Repository, there is an ingenious solution by Mr. Ivory of a problem in dynamics, first proposed by Mr. Simpson in his *Miscellaneous Tracts*, and afterwards introduced by Mr. Atwood in his *Treatise on Rectilinear Motion*, pa. 289, as a particular case of a general proposition. This gentleman's solution differing so widely from Mr. Simpson's, induced me to consider his investigation; and I find that his error proceeds from a wrong application of his own general theorem in ascertaining the ratio of the second fluxions of the quantities x and y .

Should you think the following solution of Mr. Atwood's general proposition (pa. 285, of his said treatise on rectilinear motion), from which I have easily deduced a solution to Mr. Simpson's problem; deserving of a place in your valuable work, I request you will do me the favour to publish it.

I am, Sir, your's, &c.

JOHN BARRY.

*Limerick,
July 29, 1806.*

PROPOSITION, by Mr. ATWOOD.

Let AFC, BGC (fig. 8, pl. A.), be two curves, the planes of which are vertical; and let a line, ACB, be stretched over a fixed pulley, C, by two given weights A and B, of which B preponderates against A, and descends along the curve CGB: it is required to assign the velocity of B, when it has descended through a given perpendicular altitude, and A has ascended through a given altitude in the same time?

SOLUTION, by Mr. BARRY.

According to Mr. Atwood, let BL be the perpendicular altitude through which B has descended, and let AH be the perpendicular altitude through which A has ascended in the same

same time; and let $BL = r$, $AH = q$. Suppose the weight B to describe in its descent the evanescent arc bB ; and during the same time let the other weight A rise through the arc aA . Through b and A draw bo and Al parallel to the horizon; also through b draw bn perpendicular to CB , and Am perpendicular to Ca , and let $CB = x$, $Bn = s = Am$, $Bo = \dot{r}$, $Bb = \dot{s}$, $Aa = \dot{r}$, and $al = \dot{q}$.

Now Mr. Atwood has truly found in his said treatise, pa. 286, that the whole force by which B is urged in the direction bB , is

$$= \frac{B\dot{y} - A\dot{q}}{s}, \text{ and he has also shewn in page 287 that the equi-}$$

valent mass, which being accumulated in B, when A is removed, will resist the communication of motion in the same manner as

$$\text{the mass A ascending by the force of B is } = \frac{A\ddot{r}\ddot{r}}{s\ddot{s}}.$$

Let v = the velocity of the descending weight B, w = the velocity of the ascending weight A, t = the corresponding time, and $b = 193$ inches.

Then, by the laws of motion, $\dot{s} = v \times \dot{t}$ and $\dot{r} = w \times \dot{t}$; and since the evanescent arcs aA and bB are described in the same time, we have $\ddot{s} = \dot{v} \times \dot{t}$ and $\ddot{r} = \dot{w} \times \dot{t}$; therefore, by

$$\text{substitution, } \frac{A\ddot{r}\ddot{r}}{s\ddot{s}} = \frac{Aw\dot{w}}{vv}. \text{ Consequently, the force which}$$

accelerates B is

$$= \frac{B\dot{y} - A\dot{q}}{s} \div \left(B + \frac{Aw\dot{w}}{vv} \right) = \frac{(B\dot{y} - A\dot{q})\dot{v}\dot{v}}{s(Bv\dot{v} + Aw\dot{w})}.$$

Hence, by the laws of motion,

$$v\dot{v} = b \times \dot{s} \times \frac{(B\dot{y} - A\dot{q})\dot{v}\dot{v}}{s(Bv\dot{v} + Aw\dot{w})} = b \times \frac{(B\dot{y} - A\dot{q})\dot{v}\dot{v}}{Bv\dot{v} + Aw\dot{w}}.$$

Therefore $b(B\dot{y} - A\dot{q}) = Bv\dot{v} + Aw\dot{w}$,

and taking the fluents

$$b(B\dot{y} - A\dot{q}) = \frac{Bv^2}{2} + \frac{Aw^2}{2} - d, \text{ } d \text{ being the necessary correction.}$$

$$\text{Hence } v^2 = 2by - \frac{2bAq}{B} - \frac{Aw^2}{B} + 2d.$$

It is obvious that this equation is general, whether the bodies move in right or curved lines.

COR. It is evident that $v : w :: Bb : Aa :: \dot{s} : \dot{r}$; hence $w^2 = v^2 r^2 \div \dot{s}^2$. Therefore, by substitution, we get

$$v^2 = 2by - \frac{2bAq}{B} - \frac{Av^2 r^2}{B\dot{s}^2} + 2d. \text{ And therefore}$$

$$v = \dot{s} \sqrt{\frac{2b(By - Aq) + 2dB}{B\dot{s}^2 + Ar^2}}.$$

In order to apply this general expression for the velocity of B to the solution of Mr. Simpson's problem, which Mr. Atwood expresses in the following manner, see pa. 289 of the treatise on Rectilineal Motion.

Let two equal weights, A, A, (fig. 9, pl. A.) be fastened to the extremities of a line which goes over the fixed points E and C, which are horizontal: when the line AECA is stretched by the equal weights A, A, let a body D be fixed to the middle point D, it will descend from rest at D in the direction DB, perpendicular to the horizon, at the same time elevating the weights A, A. Suppose it were required to assign the velocity of the descending weight B, when it has described any space DB.

Let ED = a, EB = y, and DB = x. Then by substituting in the expression for the velocity in the corollary, x for y, y for q, $\dot{x}$ for $\dot{s}$, and $\dot{y}$ for $\dot{r}$, we have v, or the velocity of B = $\dot{x} \sqrt{\frac{2b(Bx - 2Ay) + 2dB}{B\dot{x}^2 + 2A\dot{y}^2}}$, where 2A is written for A, because each of the weights A, A, acts in a similar manner on B. But in the present case $\dot{y} = \frac{x\dot{x}}{y}$. Hence, by substitution,

$v = y \sqrt{\frac{2b(Bx - 2Ay) + 2dB}{By^2 + 2Ax^2}}$. Let $B \div 2A = m$, and put $y^2 - a^2$ for its equal x^2 , then we have

$v = y \sqrt{\frac{2bm x - 2by + 2dm}{(m+1)y^2 - a^2}}$, which corresponds exactly with Simpson's expression for the velocity.

Since the motion commences from the point D, then v being = 0, when x = 0 and y = a, we find $d = \frac{ba}{m} - \frac{2baA}{B}$, in which case,

$$v^2 = 2b(a^2 + x^2) \times \frac{Bx - 2A \sqrt{(a^2 + x^2)} + 2aA}{B(a^2 + x^2) + 2Ax^2}.$$

If this value of v^2 be put $= 0$, we get $x = \frac{4aAB}{4A^2 - B^2} =$ the greatest distance through which the weight B can descend before its whole motion is destroyed by the weights A, A.

ARTICLE VII.

Some properties of parallelograms, with the application of them to the moments of forces. By Mr. JOHN GOUGH.

To the Editor of the Mathematical Repository.

SIR,

Mr. Gregory has given a beautiful theorem respecting the moments of forces, in his *Mechanics*; but the analytical plan of his work having induced him to derive his demonstration from the arithmetic of fines, I flatter myself the English reader will not be displeased to see this interesting proposition in the geometrical dress, which I have endeavoured to give it, in the following essay.

I am, &c.

JOHN GOUGH.

*Middleham,
August 22d, 1806.*

PROPOSITION I.

Let ABCD (fig. 10, pl. A.) be a parallelogram, and BD a diameter of it; take any point T in BD, or in BD produced, and join AT, CT. The triangle ATB shall be equal to the triangle CTB, and the triangle ATD to the triangle CTD.

Draw the diameter AC, which will be bisected by the diameter BD in the point M. Therefore, the triangle ATM is equal to the triangle CTM, the triangle ABM to CBM, and the triangle ADM to CDM; for they have equal bases, two and two, and the same altitudes (Eu. I. vi.). Therefore, by adding equals to equals, the triangle ATB is equal to the triangle CTB; and, by taking equals from equals, the triangle ATD is equal to the triangle CTD.

PROPOSITION

PROPOSITION II.

Let $ABCD$ (fig. 11, 12, 13, pl. A.) be a parallelogram of which BD is a diameter, produce AD , CD , BD , to a , c , b : from any point S which is not in the line Bb , draw SA , SB , SC , SD , to the angular points of the parallelogram; I say, if the point S , be in either of the angles ADc , CDa , which are not divided by the diameter Bb , the sum of the triangles SCD , SAD , will be equal to the triangle SBD ; but if the point S be either within the parallelogram, or in the angle aDc , the difference of the triangles SDC , SDA , will be equal to the triangle SDB .

CASE I. Let S be in the angle ADc , which is not divided by the diameter BD . Fig. 11, pl. A.

Draw ST parallel to AD or BC . Then, as S is not in the parallelogram $ABCD$, ST will meet BD produced, suppose in T ; join AT , CT . Now, because T is in BD produced, CD divides the angle TCS , and D is in the parallelogram, while T is without it; therefore the point D and the line SD are in the triangle SCT , and therefore the triangle SCT is equal to the three triangles SCD , DCT , TDS ; but the triangles SCT , SBT , are equal (Euc. 37. i.), therefore the triangle SBT is equal to the three triangles SCD , DCT , TDS , and therefore taking the triangle TDS from each, the two triangles SCD , DCT are equal to the triangle SBD . But the triangle DCT is equal to the triangle DTA (prop. i.), that is to the triangle DSA (Euc. 37. i.). Therefore the sum of the triangles SCD , DSA , is equal to the triangles SBD .

CASE II. When the point S is within the parallelogram. Fig. 12, pl. A.

Draw ST parallel to DA or BC , which will meet the diameter BD within the parallelogram; let it meet it in T , and draw AT , CT . The triangle CDS is equal to the three triangles TSC , DTS , TCD , but the triangles TSC , TSB , are equal, because of the parallels TS , CB , (Euc. 37. i.); therefore the triangle CDS is equal to the two triangles DSB , TCD . Since T is in BD , the triangles TCD , TAD , are equal (prop. i.); and since DA , TS , are parallel, the triangles TAD , ASD , are also equal (Euc. 37. i.). Therefore the triangle CSD is equal to the two triangles DSB , ASD . And therefore, the excess of the triangle CSD above the triangle ASD , is equal to the triangle DSB .

CASE III. Let S be in the angle aDc , Fig. 13, pl. A.

Make

Make aD equal to DA and cD to DC , and complete the parallelogram $aDcb$; also draw Sa , Sb , Sc . Then the excess of the triangle ScD , above the triangle SaD , is equal to the triangle SbD (case ii.). But (Euc. i. vi.) the triangle ScD is equal to the triangle SCD , the triangle SaD to the triangle SAD , and the triangle SbD to the triangle SBD . Therefore, the excess of the triangle SCD above the triangle SAD , is equal to the triangle SBD .

COR. Draw SP , SQ , SR , perpendicular to AD , DC , BD . Then since the area of a parallelogram is double the area of a triangle having the same base and altitude, what has been proved in this proposition concerning the triangles SDC , SDA , SDB , is also proved of the rectangles under DA , DC , DB , and their respective perpendiculars SP , SQ , SR .

DEFINITION 1. Let DB (fig. 12, pl. A.) represent a force in magnitude and position, from a given point S , draw SR perpendicular to DB ; and the rectangle under DB , SR , is called the moment of that force (Gregory's *Mechanics*, pa. 15, art. 31.).

DEFINITION 2. If AD , CD , (fig. 10, pl. A.) represent two forces, acting from A and C , upon the point D , and the parallelogram of forces be completed, AD , CD , are called component forces, and BD their equivalent (ibid. pa. 14, art. 29.).

DEFINITION 3. But if the force at C act in the direction DC , while that at A acts in the direction AD , the former is said to be negative to the latter.

SCHOLIUM. A negative force is converted into a positive one, thus: In CD produced (fig. 10, pl. A.), take $cD = CD$, and suppose c to be the place of a force acting upon D , in a direction cD , and its effect upon D will be equal to that of the force at C : consequently Db will be the equivalent of the forces at C , A . Hence, if the moments of the forces cD , DA , are to be referred to the point S , and if we desire to have the point S without the parallelogram of forces, take the force $DC = Dc$, then complete the parallelogram AC , and the thing is done.

PROPOSITION III.

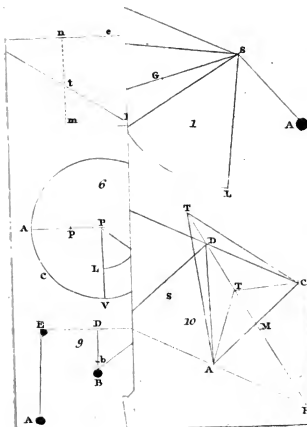
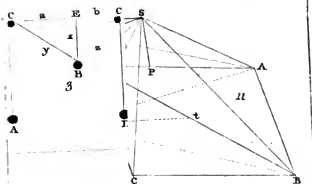
Let BA , CA , DA , EA , FA , (fig. 14, pl. A.) represent any number of forces acting upon the point A in the same plane, also let RA be their equivalent: from any point S draw Sb , Sc ,

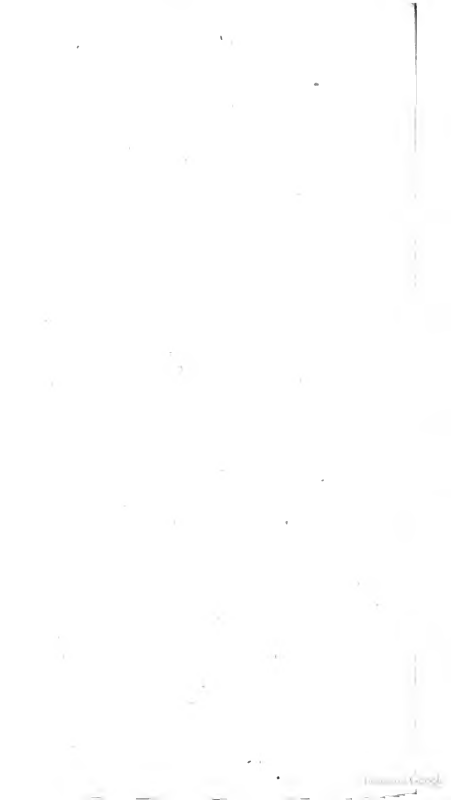
S_c, S_d, S_e, S_f, S_r , &c. perpendicular to these lines respectively. Then the moment $S_r \times AR$ will be equal to the excess of the sum of the moments $S_b \times AB, S_c \times AC, S_d \times AD$, lying on the side of AR contrary to S , above the sum of the moments $S_e \times AE, S_f \times AF$, lying on the same side of AR with S .

Let AG be the equivalent of AB, AC ; AK the equivalent of AG, AD , that is of AB, AC, AD ; and AH the equivalent of AE, AF : then since AR is the diameter of a parallelogram under AK, AH , and S is in the angle RAH , $S_r \times AR = S_k \times AK - S_h \times AH$ (prop. ii. and cor.): But $S_k \times AK = S_g \times AG + S_d \times AD$ (prop. ii.); for the same reason $S_g \times AG = S_b \times AB + S_c \times AC$, and $S_h \times AH = S_e \times AE + S_f \times AF$; therefore $S_r \times AR = S_k \times AK - S_h \times AH = S_b \times AB + S_c \times AC + S_d \times AD - (S_e \times AE + S_f \times AF)$, that is, the moment $S_r \times AR$ is equal to the excess of the sum of the moments $S_b \times AB, S_c \times AC, S_d \times AD$, lying on the side of AR contrary to S , above the sum of the moments $S_e \times AE, S_f \times AF$, lying on the same side of R with S . And what has been proved of five forces may be proved in like manner of any other number.

COR. 1. If all the component forces BA, CB, DA , and their equivalent KA , be so disposed that the point S shall not be situated in the angle KAD or that vertically opposite to it, the moment of the equivalent KA is equal to the sum of the moments of the components. This is proved in the demonstration.

COR. 2. If some of the forces in a proposed scheme be negative, they are to be converted into affirmative ones by the preceding scholium.





ARTICLE VIII.

To the Editor of the Mathematical Repository.

SIR,

The solutions of the following problems being considerably facilitated by making use of the mechanical proposition demonstrated in No. vi. (art. i. part ii. of the present volume) of your Repository, by inserting them, when it is convenient, you will oblige, Sir, Your's, &c.

A. B.

PROBLEM I.

If a body be connected with a horizontal plane by any point in it, and in such a manner, that while it descends by the force of gravity, the connected point may slide freely along the plane without friction or any obstruction whatever: it is required, for any position of the body, to investigate general expressions for the velocity of the connected point, and for the velocities of the centres of gravity and oscillation, as likewise for its pressure upon the plane.

Fig. 15, pl. B. Let QBD be the horizontal plane, Q, the point of contact at the commencement of the motion of the body, and G and P the centres of gravity and oscillation, q, g, p their position at any other time. Then (by prop. 1st. art. 2d. part 2d. of the present vol.) the point G will describe the straight line MGB, and the point P, the arc of an ellipse. Suppose the whole body, W, to be divided into two parts P and Q, and in such a manner, that $P : Q :: QG : GP$, then if the part Q be placed upon the plane at Q and connected with P by an inflexible line without weight, it is evident, that these two bodies, thus circumstanced, will move in the same manner, and with the same velocities as the corresponding points of the given body. For the distances of the centres of gravity and oscillation from Q, as well as the inclination of the line passing through them, to the horizontal plane, are the same in both cases; wherefore the above mentioned velocities must be equal.

Let $QP = L$, $QG = a$, $BG = d$, $Gg = x$, $qB = y$; v, v' and V the velocities, respectively, of Q, G and P, when they arrive at q, g , and p , and $l = 16\frac{1}{2}$ feet. It is well known that the velocities of q and g , are to each other as $Bg : qB$, hence

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d

v :

$v : v' :: d - x : y$, therefore $v = v' \times \frac{d-x}{y}$. Let fall the perpendiculars PC, pc, then by the proposition $4l \times (CP - cp) \times P = Qv^2 + PV^2$. Let pn be a small particle of the curve described by p , in a given moment of time; resolve it into pa and na , perpendicular and parallel to QC, then these lines will be as the velocities of p , in their respective directions. By the property of the centre of gravity $\frac{P+Q}{P} \cdot v' =$ velocity of P in the

the direction pa , and $\frac{Q}{P} \cdot v = \frac{Q}{P} \cdot v' \times \frac{d-x}{y} =$ its velocity in the direction na ; hence

$$\frac{(P+Q)^2}{P^2} \cdot v'^2 + \frac{Q^2}{P^2} \cdot v'^2 \cdot \frac{(d-x)^2}{y^2} = \frac{v'^2}{P^2} \times \left\{ (P+Q)^2 + Q^2 \cdot \frac{(d-x)^2}{y^2} \right\} = V^2,$$

consequently $4l(CP - cp) = 4l \cdot \frac{Lx}{a} \cdot P = Q \cdot v'^2 \times \frac{(d-x)^2}{y^2} + \frac{v'^2}{P} \times \left\{ (P+Q)^2 + Q^2 \times \frac{(d-x)^2}{y^2} \right\}$. But $P : Q :: a : L - a$,

therefore $Q = \frac{L-a}{a} \cdot P$, and $P + Q = \frac{LP}{a}$, whence, by substitution and reduction, it will easily appear that $v'^2 = 4lx \times \frac{ay^2}{(L-a)(d-x)^2 + Ly^2}$. For y^2 , in the denominator, substitute

its value $a^2 - (d-x)^2$, and we have $v'^2 = 4lx \cdot \frac{y^2}{La - (d-x)^2}$, and therefore $v' = 2\sqrt{lx} \cdot \frac{y}{\sqrt{La - (d-x)^2}}$, hence $v =$

$$\frac{d-x}{y} \cdot v' = 2\sqrt{lx} \cdot \frac{d-x}{\sqrt{La - (d-x)^2}}, \text{ and } V = 2\sqrt{lx} \cdot \sqrt{\frac{L^2a - (2L-a)(d-x)^2}{a(La - (d-x)^2)}}.$$

Put $PC = h$, then $L : a :: h : d = \frac{ah}{L}$, if therefore $x = d$, or the body has arrived at the situation $q'BD$, then $v = 2\sqrt{l \times \frac{ah}{L}} \times \sqrt{\frac{L^2a}{a^3L}} = 2\sqrt{lh}$, the same velocity which a body would acquire in descending freely through the space PC. But when $x = d$, $y = a$, therefore, when the body is in the

the above mentioned situation, $v' = 2\sqrt{l \times \frac{ah}{L}} \times \frac{a}{\sqrt{la}} = 2\sqrt{\frac{h}{l}} \times \frac{a}{L}$, and it is evident then that $v = 0$.

When x is very small or evanescent, let $y = c$, then we have $v^2 = 4lx \cdot \frac{c^2}{La - d^2}$; but when the body and space descended are given, the accelerating force varies as the square of the velocity generated, wherefore $4lx : 4lx \cdot \frac{c^2}{La - d^2} :: W(P+Q) : W \times \frac{c^2}{La - d^2} =$ that part of the weight of W which is not supported; hence, $W \times \left(s - \frac{c^2}{La - d^2}\right) =$ pressure upon the plane.

Cor. Let s and $c =$ sine and cosine of the angle QGB for any position of the body whatever, then $y = sa$, and $d - x = ca$, rad. $= 1$, wherefore, by substitution, $v' = 2\sqrt{(lx) \times \frac{sa}{\sqrt{la - c^2a^2}}} = 2\sqrt{(lx)} \times s \sqrt{\frac{a}{L - c^2a}}$, $v = 2\sqrt{(lx)} \times \frac{ca}{\sqrt{(La - c^2a^2)}} = 2\sqrt{(lx)} \times c \sqrt{\frac{c}{L - c^2a}}$ and $V = 2\sqrt{(lx)} \times \sqrt{\frac{L^2 - (2L - a) \cdot c^2a}{La - c^2a}}$. In the very same manner it will appear, that the pressure upon the plane $= W \times \left(1 - \frac{s^2a^2}{La - c^2a^2}\right) = W \times \left(1 - \frac{s^2a}{L - c^2a}\right) = W \times \left(1 - \frac{s^2a}{(L - a) - s^2a}\right)$. It is evident that when the body descends below the plane, c , will be negative, or the point Q will move in a contrary direction to what it did before.

Ex. 1. Suppose the body to be a slender cylinder or rod; then if $R =$ its length, it is well known that $a = \frac{1}{2}R$ and $L = \frac{1}{2}R$, therefore $a = \frac{1}{2}L$; hence, by substituting for a , its value, in the cor. it will easily appear that $v' = 2\sqrt{(lx)} \cdot s \sqrt{\frac{3}{4 - 3c^2}}$, $v = 2\sqrt{(lx)} \cdot c \sqrt{\frac{3}{4 - 3c^2}}$ and $V = 2\sqrt{(lx)} \times \sqrt{\frac{16 - 15c^2}{12 - 9c^2}}$. Hence when the rod arrives at the plane, and consequently,

d 2

s = 1,

$s = 1$, $c = 0$, and $d = x$, we have $v' = 2\sqrt{(ld)} \times \sqrt{\frac{1}{2}} = \sqrt{(3ld)}$, $v = 0$, and $V = 2\sqrt{(ld)} \times \sqrt{\frac{1}{2}} = 4\sqrt{\frac{ld}{3}}$. Likewise the pressure upon the plane $= W \times \left(1 - \frac{s^2 a}{L - c^2}\right) = W \times \left(1 - \frac{\frac{1}{2}s^2 L}{L - \frac{1}{4}c^2 L}\right) = W \times \left(1 - \frac{3s^2}{4 - 3c^2}\right) = W \times \left(1 - \frac{3s^2}{1 + 3s^2}\right) = W \times \frac{1}{4}$ when $s = 1$ and $c = 0$.

Ex. 2. Prize Question, Gentleman's Diary, 1791.

If a given segment of a globe greater than an hemisphere touching an horizontal plane perfectly polished, with its surface at the edge of its section or base, be thus put in motion by the force of uniform gravity; how far will it move along the plane, what is its velocity, and with what weight does it press against the plane at every point during the time of one titubation?

Let SQP_g (fig. 16, pl. B.) represent the position of the segment when its motion commences, SP , QT , two diameters of the globe perpendicular to and parallel to the horizon; draw RC perpendicular to Sg and produce it to O , the centre of oscillation of the segment, when C , the centre of the globe, is the point of suspension. Let G be the centre of gravity of the same segment, and through G , draw Gd perpendicular to the horizon, it is evident that the point G will descend in this line, and that the point C will move in QT .

Put $SC = r$, $CR = b$, $CG = a$, $Ga = d$ and $CO = L$. By similar triangles we have, $r : b :: a : d = \frac{ab}{r}$, hence, by cor. $v' = 2\sqrt{(lx)} \times s\sqrt{\frac{a}{L - c^2 a}}$, $v = 2\sqrt{(lx)} \times c\sqrt{\frac{a}{L - c^2 a}}$ and $V = 2\sqrt{(lx)} \times \sqrt{\frac{L^2 - (eL - a) \cdot c^2 a}{a \cdot (L - c^2 a)}}$; likewise the pressure upon the plane $= W \times \left(1 - \frac{s^2 a}{L - c^2 a}\right)$.

When G comes to a , then $s = 1$, $c = 0$, and $x = d = \frac{ab}{r}$, therefore in that position, $v' = 2\sqrt{\left(l \times \frac{ab}{r}\right)} \times \sqrt{\frac{a}{L}}$ $= 2s\sqrt{\frac{lb}{rL}}$, $v = 0$, $V = 2\sqrt{\left(l \times \frac{ab}{r}\right)} \times \sqrt{\frac{L}{a}} = 2\sqrt{\frac{lLb}{r}}$, and the pressure $= W \times \left(1 - \frac{a}{L}\right)$.

When

When C arrives at a , in that position we shall have $s = a$, $c = 1$, $x = a + \frac{ab}{r} = ar(r+b)$, and then it will appear that $v' = 0$, $v = 2\sqrt{\left\{l \cdot \frac{a}{r} \cdot (r+b)\right\} \times \sqrt{L-a}} = 2a\sqrt{\frac{l(r+b)}{r(L-a)}}$, $V = 2\sqrt{\left\{l \cdot \frac{a}{r} \cdot (r+b)\right\} \times \sqrt{\frac{L^2 - (2L-a) \cdot a}{(L-a) \cdot a}}}$, and the pressure upon the plane $= W \times 1$. The values of a and L are easily found by the usual methods of calculation.

It may be observed, that when N comes to P, or its greatest altitude; the rotatory motion of the segment will be the greatest possible, but it will continue to move the same way till it arrives at N', being at the same distance from the horizontal plane as when it began, when it will entirely cease to move; it will then begin to move the contrary way, and so continue to vibrate for ever. The space passed over by the centre of the globe, forwards and backwards, in one titubation is evidently $= 2CG$, the greatest distance on each side of a , being $= CG$.

In the very same manner, by a proper substitution for L and a , the values of v' , v and V may be found for any body whatever; as likewise its pressure upon the plane, its position, or the values of s and c , being given.

PROBLEM II.

Prop. xxiv. lect. vi. Atwood on Rectilinear and Rotatory Motion.

Let A, B (fig. 17, pl. B.), be a single moveable and a fixed pully, by means of which the power P elevates the weight W: having given P and W, together with the weight of the cylindrical pullics, A and B, it is required to assign the space which the descending weight P describes in a given time, the weight of the moveable pully being included in the weight W.

Let $v =$ velocity of P, then $\frac{v}{2} =$ velocity of W, $\frac{v}{2\sqrt{2}} =$ velocity of the centre of gyration of the pully A, and $\frac{v}{2\sqrt{2}} =$ velocity of the same centre in the pully B. Let $s =$ space descended by P in the time t ; then we have the following equation, viz.

$$4(Ps - W \cdot \frac{s}{2}) = P v^2 + W \cdot \frac{v^2}{4} + Q \cdot \left(\frac{v^2}{2} + \frac{v^2}{8} \right) = P v^2 + W \cdot \frac{v^2}{4} + Q \cdot \frac{5v^2}{8},$$

but

but $v = \frac{2s}{t}$, therefore $4l \left(Ps - W \cdot \frac{s}{2} \right) = \frac{4P^2 s^2}{t^2} + W \cdot \frac{s^2}{t^2} + Q \cdot \frac{5s^2}{2t^2}$, therefore $s = \frac{4lt^2 \times (2P - W)}{8P + 2W + 5Q}$.

N.B. Q equal the weight of each pully.

PROBLEM III.

Prop. xxv. sect. vi. Atwood's Treatise on Motion.

In a system of pullies in which the same string goes round all the pullies contained in two blocks: having given the power P and the weight W raised by it, together with the number of the pullies and the weight and figure of each, it is required to assign the force which accelerates the descent of P , the weight of the lower block being included in the weight W .

Let t = any time, v = velocity generated in P in that time, and s = space descended; then by mechanics $\frac{v}{n}$ = velocity of W . The velocity of the circumference of the slowest pully = that of $W = \frac{v}{n}$, the velocity of the circumference of the next slowest = $\frac{2v}{n}$, of the next, = $\frac{3v}{n}$, &c. to $\frac{nv}{n}$; and the velocities of their respective centres of gyration = $\frac{v}{n\sqrt{2}}$, $\frac{2v}{n\sqrt{2}}$, $\frac{3v}{n\sqrt{2}}$... $\frac{nv}{n\sqrt{2}}$. Hence this equation $4l \left(Ps - W \cdot \frac{s}{n} \right) = P v^2 + W \cdot \frac{v^2}{n^2} + Q \cdot \left\{ \frac{v^2}{2n^2} \times (1 + 4 + 9 \dots n^2) \right\}$. For s , substitute $\frac{vt}{2}$, then $2lt \times (12Pn^2 - 12Wn) = \left\{ 12Pn^2 + 12W + Q \times (2n^2 + 3n^2 + n) \right\} \times v$; therefore $v = 2lt \times \frac{12Pn^2 - 12Wn}{12Pn^2 + 12W + Q \cdot (2n^2 + 3n^2 + n)}$.

Make $t = 1$, then this expression will represent the force which accelerates P , when the force of gravity = $2l$.

N.B. The sum of $1 + 4 + 9 \dots n^2 = \frac{1}{3}(2n^2 + 3n^2 + n)$.

Q = the weight of each pully, which are supposed cylindrical.

ARTICLE

ERRATA.

In page 4, vol. 2, part 2, line 11, from the bottom,

for $A \times \frac{x^2 u^2}{y^2} + D \times \frac{x^2 u^2}{z^2}$, read $A \times \frac{x^2 u^2}{y^2} + B u^2 + D \times \frac{x^2 u^2}{z^2}$

ARTICLE IX.

A DIOPHANTINE PROBLEM. *By Mr. CUNLIFFE.*

To find values for the sides of a triangle in rational numbers such, that the lengths of three right lines from the angles to the middle of the opposite sides may be expressed by rational numbers.

SOLUTION.

Let ACB (fig. 18, pl. B.) represent the triangle, CD, AE and BF right lines from the angles to the middle of the opposite sides: put $AD = DB = x$, $AF = FC = y$, and $BE = EC = z$. Then by a known property of triangles $AC^2 + BC^2 - 2AD^2 = 2CD^2$, or $\frac{1}{2}(AC^2 + BC^2) - AD^2 = 2y^2 + 2z^2 - x^2 = CD^2 = \text{a square.}$

In like manner $2x^2 + 2z^2 - y^2 = BF^2 = \text{a square;}$

and $2x^2 + 2y^2 - z^2 = AE^2 = \text{a square.}$

Put $x + y - n = z$, then $2x^2 + 2y^2 - z^2 = 2x^2 + 2y^2 - (x + y - n)^2 = x^2 - 2xy + y^2 + 2n(x + y) - n^2 = \text{a square;}$ and $2x^2 + 2z^2 - y^2 = 2x^2 + 2(x + y - n)^2 - y^2 = 4x^2 + 4xy + y^2 - 4n(x + y) + 2n^2 = \text{a square.}$

Assume $x - y + s$ for the root of the first, and $2x + y - r$ for the root of the second of the preceding expressions: then, $x^2 - 2xy + y^2 + 2n(x + y) - n^2 = (x - y + s)^2 = x^2 - 2xy + y^2 + 2s(x - y) + s^2$; and $4x^2 + 4xy + y^2 - 4n(x + y) + 2n^2 = (2x + y - r)^2 = 4x^2 + 4xy + y^2 - 2r(2x + y) + r^2$; from the former of these

$$x = \frac{n^2 + s^2 - 2y(n + s)}{2(n - s)} = \frac{2n^2 + 2s^2 - 4y(n + s)}{4(n - s)},$$

$$\text{and from the latter } x = \frac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)};$$

$$\text{therefore } \frac{2n^2 + 2s^2 - 4y(n + s)}{4(n - s)} = \frac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)},$$

making the numerators and denominators equal to each other, $2n^2 + 2s^2 - 4y(n + s) = r^2 - 2n^2 + 2y(2n - r)$,

$$\text{from whence } y = \frac{4n^2 + 2s^2 - r^2}{8n + 4s - 2r}.$$

$$\text{And } 4(n - s) = 4(r - n),$$

$4(r - n)$; whence $n = \frac{1}{2}(r + s)$, by means of which
 $y = \frac{6s^2 + 4rs}{4(r + 4s)}$; $x = \frac{n^2 + s^2 - 2y(n + s)}{2(n - s)} = \frac{r^2 + 3rs - 2s^2}{4(r + 4s)}$;
 and $z = x + y - n = \frac{-(4s^2 + 3rs + r^2)}{4(r + 4s)}$.

Now as only the squares of x , y , and z are concerned in the problem, it is manifest that either positive or negative values of these quantities may be taken; wherefore, rejecting the common denominator $4(r + 4s)$, we may put $x = r^2 + 3rs - 2s^2$, $y = 6s^2 + 4rs$, and $z = r^2 + 3rs + 4s^2$, and these being written in the remaining expression to be made a square, viz. $2y^2 + 2z^2 - x^2$ it becomes $r^4 + 6r^3s + 61r^2s^2 + 156rs^3 + 100s^4$.

Assume $r^2 = \frac{39rs}{5} - 10s^2$ for its root, that is, put

$$r^4 + 6r^3s + 61r^2s^2 + 156rs^3 + 100s^4 = (r^2 - \frac{39rs}{5} - 10s^2)^2 \\ = r^4 - \frac{78r^3s}{5} + (\frac{39}{5})^2 \times r^2s^2 - 20r^2s^2 + 156rs^3 + 100s^4;$$

which, after proper reduction gives, $r = -\frac{14s}{15}$.

Take $s = 15$, then $r = -14$, $x = -884$, $y = 510$ and $z = 466$; and as these values are all even numbers, they will express the sides of a plane triangle that will answer.

Let us now examine the result of a general method of proceeding. In the preceding part of the solution we obtained the equation

$$\frac{2n^2 + 2s^2 - 4y(n + s)}{4(n - s)} = \frac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)};$$

from whence we have $y = \frac{-2n^2(r - s) + n(r^2 + 2s^2) - rs(r + 2s)}{2\{n(4s - r) - 3rs\}}$,

and by means of this

$$x = \frac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)} = \frac{n^2(r + 2s) - n(r^2 + 2s^2) - rs(r - s)}{2\{n(4s - r) - 3rs\}};$$

$$\text{and } z = x + y - n = \frac{-n^2(4s - r) + 6rsn - rs(2r + s)}{2\{n(4s - r) - 3rs\}}.$$

And as only the squares of x , y , and z are concerned in the problem, it is evident that either positive or negative values of these quantities may be taken; therefore rejecting the common denominator $2\{n(4s - r) - 3rs\}$ we may take

$$x = n^3(r + 2s) - n(r^2 + 2s^2) - rs(r - s),$$

$$y = 2n^3(r - s) - n(r^2 + 2s^2) + rs(r + 2s),$$

$$z = n^3(4s - r) - 6rsn + rs(2r + s);$$

and these values being written in the remaining expression to be made a square, viz. $2y^2 + 2z^2 - x^2$, it becomes

$$9n^4(2s-r)^2 + 2n^3(12s^3 - 54s^2r + 18sr^2 - 3r^3) + n^2(r^4 - 34r^3s + 150r^2s^2 + 68rs^3 + 4s^4) + 2n(3r^4s - 21r^3s^2 - 6r^2s^3 + 6rs^4) + 9r^2s^2(r+s)^2 = \text{a square.}$$

The foregoing expression might manifestly be made a square, generally, by well known methods, but the calculation would be tedious.

It may be observed, that in the preceding general expressions for x , y , and z , any two of the quantities n , r , and s , may be taken at pleasure, by which means the solution in particular cases will be rendered much more simple.

Example. Take $s = 1$ and $r = -1$, then $x = n^3 - 3n - 2$, $y = -4n^3 - 3n - 1$, or, $y = 4n^3 + 3n + 1$, and $z = 5n^3 + 6n + 1$; and hence $2y^2 + 2z^2 - x^2 = 81n^4 + 174n^3 + 121n^2 + 24n = \text{a square.}$

Assume $9n^3 + \frac{29n}{3}$ for the root; then $81n^4 + 174n^3 + 121n^2 + 24n = (9n^3 + \frac{29n}{3})^2 = 81n^4 + 174n^3 + n^2(\frac{29}{3})^2$, which after proper reduction gives $n = \frac{-27}{81}$; whence $x = \frac{1318}{961}$, $y = \frac{1366}{961}$, and $z = -\frac{416}{961}$.

Rejecting the common denominator 961, we may take $x = 1318$, $y = 1366$, and $z = 416$: and as these are all even numbers they will denote the sides of a triangle that will answer.

Again, put $m - 1 = n$, then $81n^4 + 174n^3 + 121n^2 + 24n = 81m^4 - 150m^3 + 85m^2 - 20m + 4 = \text{a square} = (9m^2 - \frac{25m}{3} + 2)^2 = 81m^4 - 150m^3 + \frac{625}{9}m^2 + 36m^2 - \frac{100}{3}m + 4$, from whence $m = \frac{15}{23}$, $n = m - 1 = -\frac{8}{23}$, $x = \frac{442}{529}$, $y = \frac{239}{529}$, and $z = \frac{255}{529}$.

Rejecting the common denominator 529, and doubling the numerators to make them even numbers, we shall have 884, 466, and 510 for the sides of a triangle that will answer, being the same as were found in the first part of the solution.

From what has been done, it will be plain how other answers may be obtained, if any one thinks it worth the trouble of calculation.

It will be recollected that there is another solution to the foregoing question vol. i, part ii, art. x : but besides some difference in the methods of solution, one of the sets of numbers here found is considerably smaller than those found in the article alluded to.

ARTICLE X.

THE THEORY OF AMICABLE NUMBERS.

To the Editor of the Mathematical Repository.

SIR,

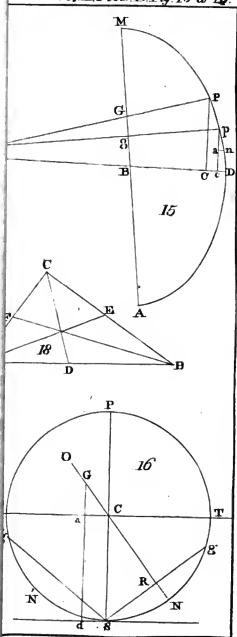
The Theory of Amicable Numbers is a subject which has been almost overlooked by the Mathematicians of this Country; and for any thing I know to the contrary, the mere English reader might have remained ignorant, both of the name and nature of this species of numbers, had not Dr. Hutton briefly explained their properties in the first vol. of his *Mathematical Dictionary*. The Article, here alluded to, is professedly taken from a Treatise, by F. Van Schooten, entitled "*Ratio inveniendi numeros amicabile*," or the method of finding amicable numbers. It is certain that this writer borrowed his knowledge of the numbers in question from Des Cartes; for he has preserved a rule for finding them, invented by this great mathematician. As for his own enquiries they are confined to particular cases, his intention being to discover a series of amicable numbers increasing progressively; and this appears to be the reason, why he gives the Cartesian rule unaccompanied by a demonstration. I flatter myself that the following essay will be found to contain a more comprehensive view of the subject, derived from considerations of a more general nature: at the same time I have not the vanity to pronounce it a complete Treatise. The piece comprises several limits and general properties relating to Amicable Numbers; which may lead to a perfect Theory in abler hands. Perhaps this may have been done by some of the Foreign Mathematicians; but Van Schooten is the only writer on the subject who has come to the knowledge of

Your's, &c.

JOHN GOUGH.

*Middleshaw, near Kendal,
March 19, 1807.*

Article



Article 1. Definition. When two numbers are so constituted, that each of them is equal to the sum of the divisors of the other, they are called Amicable Numbers, which name was probably invented by Van Schooten.

Example. 284 and 220 are Amicable Numbers; for the sum of the divisors of 220 = $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$; also the divisors of 284, taken collectively, = $1 + 2 + 4 + 71 + 142 = 220$.

2. Let ax and ayz be two Amicable Numbers consisting of a common measure, a , multiplied by the primes x , y , and z , to find the relations of these primes.

Put q = the sum of the divisors of a ; then

$$ax = a + q + (a + q)y + (a + q)z + qyz \text{ (art. 1.)};$$

for the same reason $ayz = a + q + qx$.

Invert the latter equation, and add it to the former, and

$$(a + q)(1 + x) = (a + q)(1 + y + z + yz):$$

Hence $(1 + x) = (1 + y)(1 + z)$; and $x = (y + z) + yz$.

3. Suppose adx , $aryz$, &c. to represent a pair of Amicable Numbers, consisting of a common measure a , combined with a greater number of primes than those of the form in art. 2, and by art. 1, we shall have the two following equations,

$$(a + q)(1 + d + x) + qdx = aryz, \text{ and}$$

$$adx = (a + q)(1 + r + y + z + ry + rz + yz) + qryz:$$

$$\text{Hence } (1 + d)(1 + x) = (1 + r)(1 + y)(1 + z).$$

And the same may be proved of any other number of primes.

Hence it appears, that if a pair of Amicable Numbers be divided by their greatest common measure, and the prime divisors of these quotients be severally increased by unity, the products of the two sets thus augmented, will be equal.

4. If ax , ayz be a pair of Amicable Numbers, we have

$$a + q + qx = ayz \text{ (art. 1.)}.$$

Hence, as $a : q :: 1 + x : yz - 1 :: (1 + y)(1 + z) : yz - 1$ (art. 2.).

Again, if adx , $aryz$, &c. be a pair of Amicable Numbers, we have $(a + q)(1 + d + x) + qdx = aryz$ (art. 3.).

Hence, as $a : q :: (1 + d)(1 + x) : ryz - (1 + d + x) :: (1 + r)(1 + y)(1 + z) : ryz - (1 + d + x)$ (art. 3.), which may also be proved of any other number of primes.

Hence, if a be given, q is given, but q must be less than a ; and if two sets of primes, namely, d , x , and r , y , z can be found, which will make $(1 + d)(1 + x) = (1 + r)(1 + y)(1 + z)$, &c.

and also give the following proportion, as $a : q :: (1+d)(1+x) : xyz - (1+d+x)$, then will $adx, aryz$ be Amicable Numbers, which is a general property. But the sequel of this essay will be confined to the Cartesian form, ax, ayz .

5. Since $q+qx=ayz-a$, (art. 4.) $= qy+qz+qyz$ (art. 2.), we have $z = \frac{a+q+qy}{ay-qy-q}$, and $z+1 = \frac{a+ay}{ay-qy-q}$;

for the same reason $y = \frac{a+q+qz}{az-qz-q}$, and $y+1 = \frac{a+az}{az-qz-q}$;

but $x = y+z+yz$ (art. 2.) $= \frac{a+q+ay+qy+ay^2}{ay-qy-q} = \frac{a+q+az+qz+az^2}{az-qz-q}$; hence if a , and either of the

primes y or z be given, the other with x may be found, if the data will allow them to be primes; also, we have

$$x+1 = \frac{\{a(1+y)\}^2}{y-qy-q} = \frac{\{a(1+z)\}^2}{az-qz-q}.$$

6. Since $(1+x)=(1+y)(1+z)$ (art. 2.) also $(1+y) = \frac{a+az}{az-qz-q}$,

and $(1+z) = \frac{a+ay}{ay-qy-q}$ (art. 5.), we have by multipli-

cation and substitution $(1+x) = \frac{a^2+a^2x}{(az-qz-q)(ay-qy-q)}$;

therefore $(az-qz-q)(ay-qy-q) = a^2$; consequently $(az-qz-q)$ and $(ay-qy-q)$ are severally divisors of a^2 .

7. Let f be a divisor of a^2 and put $ay-qy-q = f$; then $y = \frac{f+q}{a-q}$; but $az-qz-q = \frac{a^2}{ay-qy-q}$ (art. 6.) $= \frac{a^2}{f}$; and

$z = \frac{a^2+fq}{af-fq}$; hence $y+1 = \frac{a+f}{a-q}$, and $z+1 = \frac{a^2+af}{af-fq}$;

Hence, as $y+1 : z+1 :: f : a$.

8. Besides the relations which have been shewn to obtain amongst the factors of Amicable Numbers, other conditions and limits remain to be pointed out. In the first place then, no two of the primes x, y , and z can be equal, or in other words, x, y , and z must be three different numbers. For if they be not, y must be equal to z or x : first, put $y=z$, then $x = y+z+zy$, that is, $x = 2y+y^2$, or a prime = to a composite number, which is absurd. Again, put $y=x$, then $x+1 = (y+1)(z+1) = (x+1)(z+1)$; hence $(z+1) = 1$, and $z = 0$, which is also absurd; therefore x, y , and z are different primes.

9. No one one of the three primes x, y, z can be equal to 2,

If the contrary be maintained, the absurdity of the supposition may be proved thus: let y be less than x or z ; then if 2 be one

of the three primes, $y = 2$; but $\frac{ay + a}{ay - qy - q} = z + 1$ (art. 5.)

$\equiv$ an even number; that is, $\frac{3a}{2a - 3q} \equiv$ an even number;

therefore $a \equiv$ an even number; consequently a may be represented by $2p, 4p, \&c. 2pq, 4pq, \&c.$ where $p, q, \&c.$ are primes; put $a = 2p$; then the Amicable Numbers are $2p \times x$, and $2p \times 2z = 4p \times z$, and we have (by art. 2.) $1 + x = (1 + 2)(1 + z) = 3 + 3z$: but by collecting the divisors of each number, as in art. 1, we get $1 + 2 + p + 2p + x + 2x + px = 3 + 3p + 3x + px = 4pz$; and $1 + 2 + 4 + 2p + 4p + z + 2z + 4z + pz + 2pz = 7 + 7p + 7z + 3pz = 2px$; hence, by adding the equations, we get $7 + 7p + 7z + 7pz = 3 + 3p + 3x + 3px$, that is, $(7 + 7p)(1 + z) = (3 + 3p)(1 + x)$: but we have shewn that $1 + x = 3 + 3z$; consequently $(7 + 7p)(1 + z) = (9 + 9p)(1 + z)$, an absurdity; therefore 2 cannot be substituted for x, y , or z .

10. a is not a prime. For if a be a prime, $q = 1$, and f , in art. 7, $= a$ or 1: if $f = a$, we have $y + 1 : z + 1 :: a : a$ by that art.; that is $y = z$, which is impossible (art. 8.): again,

let $f = 1$, and $y = \frac{f + q}{a - q}$; that is $y = 2$ or 1, but y cannot be $= 2$ (art. 9.), and 1 is not a number; hence a is not a prime.

11. In the next place it is necessary to find the limits of the value of q relative to a , which may be done thus: it is evident from the proportion in art. 4, that q is less than a ; but to find a part of a which is less than q , proceed as follows. We have

an affirmative number $z + 1 = \frac{ay + a}{ay - qy - q}$ (art. 4.), but

z is odd (art. 9); consequently $z + 1$ may be expressed by $2n$

an even number; hence $y = \frac{2nq + a}{2na - a - 2nq}$: but y is a whole

number; therefore $2nq + a$ is greater than $2na - (a + 2nq)$: to each add $2nq - a$, and $2nq$ is greater than $na - a$; therefore

$\frac{2n}{n - 1}$ will be greater than $\frac{a}{q}$; but $\frac{2n}{n - 1}$ cannot exceed 4;

consequently q is greater than $\frac{a}{4}$.

12. If a , be a power of a prime r , namely $a = r^n$, then $q = \frac{a-1}{r-1}$. For $q = 1 + r + r^2 + \dots + r^{n-1}$. Hence, if $r = 2$, $q = a - 1$, and, if $r = 3$, $q = \frac{1}{2}(a - 1)$.

13. Let $a = r^n$, r being a prime; and $q = \frac{a-1}{r-1}$ (art. 12), which is greater than $\frac{a}{4}$ (art. 11.); therefore $4a - 4$ is greater than $ra - a$: hence $(5 - r)a$ is greater than 4 : therefore a is greater than $\frac{4}{5-r}$: but a is affirmative; consequently the prime $r = 2$ or 3 . Put $r = 3$, and $q = \frac{1}{2}(a - 1)$ (art. 11): hence $a - q = \frac{a+1}{2}$; consequently $z + 1 = \frac{2ay + 2a}{ay + y + 1 - a}$; but z is an odd number greater than 3 ; therefore $z - 1 = \frac{4a - (2 + 2y)}{ay + y + 1 - a} =$ an even number $= 2m$, and $y = \frac{ma + 2a - (1 + m)}{ma + 1 + m}$, in which expression the denominator is greater than half the numerator, that is, y is a fraction less than 2 , which is absurd; consequently a is a power of 2 , universally in the Amicable Numbers of Des Cartes.

14. Since $a = 2^n$, in which expression n is greater than 1 (by art. 10, 13.); and f , in art. 7, is less than a , and a divisor of a^2 , it follows, that f is also a divisor of a ; consequently we may write pf for a , and $pf - 1$ for q ; hence $y = f + q = f + pf - 1$ (art. 7.), and $y + 1 = f + pf$: but as $y + 1 : z + 1 :: f : pf$; therefore $z + 1 = pf(p + 1)$, and $x + 1 = (y + 1)(z + 1) = pf^2(p + 1)^2$: now if the three values of $y + 1$, $z + 1$, and $x + 1$, severally diminished by unity, happen to be primes, the expressions $pfx = ax$, and $pfyz = ayz$ will constitute a pair of Amicable Numbers.

15. Put $p = 2$, then $y = 3f - 1$, $z = 6f - 1$, $x = 18f^2 - 1$, which is the substance of the rule given by Des Cartes, (vid. Francisci a Schooten Math. Exercitationes, Leyden, pa. 423, or Hutton's Dictionary, vol. 1, pa. 105).

But there are strong reasons to believe that Amicable Numbers of the form ax , and ayz exist, which cannot be detected by this rule, namely, when $p = 2^n$, n being greater than unity. In these cases, however, x, y , and z will be high numbers, particularly the

the first; and few persons would be willing to undertake the trouble of determining whether they are or are not primes: I shall therefore conclude this essay with the following observation.

16. If the primes x , y , and z be given making $x + 1 = (y + 1)(z + 1)$, to find if they can constitute a pair of Amicable Numbers. Divide $z + 1$ by $y + 1$, and call the quotient p ; then, if p be not 2 or a power of 2, the thing is impossible: but if p be some power of 2, divide $y + 1$ by $p + 1$, and put the quotient $= f$; then, if f be not 2 or a power thereof, the thing is impossible: but if $f = 2^n$, the common multiplier $a = pf$.

ARTICLE XI.

A New Solution of a Problem in Insurance of Money on Lives: in which it is demonstrated, that the tables for that purpose now in use are deficient by about a 67th. part of the whole.

By PHILALETES CANTABRIGIENSIS.

1. The late increase of the number of offices in London for granting annuities, and insuring sums of money, on lives, has occasioned my resuming a subject which I had laid aside for many years.

Such offices, (when their number is not too great, and) when their tables are constructed on right principles, and their affairs are conducted with integrity, must prove very beneficial to the public, at the same time that the proprietors derive from them a reasonable profit; but, when their calculations are erroneous, either the public must be injured by paying too much, or the proprietors must, in time, be ruined by receiving too little: and, in the latter case, many of the purchasers, or their widows or children, must be deprived of that benefit which they expected.

On a perusal of the works of the latest and most celebrated writers on this branch of the mathematics, my surprise has been excited at finding that many of their calculations are grounded on an erroneous principle; (viz. taking that to be annual chance which is momentary, and allowing as much discount of money on the value of the chance in the first moment of the year as in the last moment of it :) in consequence of which the results are
deficient

deficient by about a 67th. part of the whole. This, I doubt not, will appear to every competent judge of these matters, who shall peruse the solution of the following problem. I have only to add, by way of preface, that, although I have never had any personal acquaintance with the writers above alluded to, and am under no sort of obligation to them, yet I omit their names; supposing that the cause of truth will be sufficiently served in this instance by the calculations and observations which here follow.

PROBLEM.

2. A person of a given age, and in good health, is desirous of insuring to his executor a sum of money denoted by S , to be paid at his death, whenever that may happen: it is required to find P , the present value of that sum, according to any table of the probable duration of human life, and at any given rate of interest.

SOLUTION.

3. This is evidently a momentary chance; and although no tables of the probable duration of human life, that I know of, have been published for shorter periods than years, yet a solution accurate enough for common use may be obtained from them. It will be sufficient also, in most cases, to consider the decrements of life as uniform *for the space of a year*, at all ages: above 7 years. Lastly, all the accuracy requisite in these matters may be obtained without the use of fluxions, by dividing the year into seconds, and using common algebra.

4. These things being premised, let the number of persons living at the given age be denoted by a , and the number living at the ages which exceed the given one by 1, 2, 3, 4, and 5 years, by b , c , d , e , and f , respectively. Let q denote the interest of £1 for one year, and $R (= 1 + q)$ the amount of £1 in a year. Let m denote the number of seconds in a year, and r the amount of £1 in one second, i. e. let $r^m = R = 1 + q$.

Now, since the decrements of life are supposed to be uniform during the year, and since the number of persons of the given age that die in the whole of the first year is $a - b$, the number that will die in the m th part of that year will be $\frac{a - b}{m}$; and the chance which the person has to die in that part of the year, whether it be taken at the beginning, at the middle, or at the end of

of it, will be expressed by $\frac{a-b^m}{am}$. The number of these equal chances in the whole year is evidently $= m$; and the present value of the sum S , to be received on all these chances, is the sum of the series $\frac{a-b}{amr}S + \frac{a-b}{amr^2}S + \frac{a-b}{amr^3}S, \&c.$ continued to m terms, $= \frac{a-b}{am}S \times \left(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \&c. \text{ to } \frac{1}{r^m} \right)$
 $= \frac{a-b}{am}S \times \left(\frac{1}{r-1} - \frac{1}{r^m \times (r-1)} \right)$; which expression, by writing R for its equal, r^m , and reducing the fractions to a common denominator, becomes $\frac{a-b}{am}S \times \frac{R-1}{R(r-1)} = \frac{a-b}{aR}S \times \frac{R-1}{m(r-1)}$.

5. Again, since the number of persons that die in the second year is $b-c$, the chance of one person's dying during that whole year is $\frac{b-c}{a}$, and of his dying in any m th part of it will be denoted by $\frac{b-c}{am}$; and the present value of receiving the sum S on m such chances will be $= \frac{b-c}{amRr}S + \frac{b-c}{amRr^2}S + \frac{b-c}{amRr^3}S, \&c. \text{ to } m \text{ terms, the sum of which (by the preceding art.) is } = \frac{b-c}{amR}S \times \frac{R-1}{R(r-1)} = \frac{b-c}{aR}S \times \frac{R-1}{m(r-1)}$.

In

* This may be illustrated, for the satisfaction of those who are not well acquainted with the doctrine of chances, as follows: According to Dr. Halley's observations, out of 427 persons living at the age of 42 years, 10 die in the space of 1 year. Therefore, by the hypothesis, 1 will die in the 10 part of a year; and the chance which a person of the aforesaid age has to live the first 10th part of a year is expressed by the fraction $\frac{417}{427}$, certainly being denoted by 1; and the chance which he has to die in the first 10th part of the year is expressed by the fraction $\frac{10}{427}$. Again, since his chance for living to the end of the first 10th part of the year is expressed by $\frac{417}{427}$; and out of 426 persons living at that age 1 will die in the next 10th part of the year, the chance which he has to live to the beginning of the second 10th part of a year and to die before the end of it, is expressed by $\frac{417}{427} \times \frac{1}{426} = \frac{1}{427}$. And, by a similar argumentation, it will appear that the chance of the persons dying in any one 10th part of the year will be expressed by the fraction $\frac{1}{427}$. And, in like manner, the chance which the same person has to die

in 100th.	} part of that year will	{	$\frac{1}{42700}$
in 1000th.			$\frac{1}{427000}$
in 10000th.			$\frac{1}{4270000}$
&c.			&c.
		be expressed by	

In like manner, the present value of all the chances of receiving the sum S , during the third, fourth, and fifth years, will be found to be $\frac{c-d}{aR^3} S \times \frac{R-1}{m(r-1)}$, $\frac{d-e}{aR^4} S \times \frac{R-1}{m(r-1)}$, and $\frac{e-f}{aR^5} S \times \frac{R-1}{m(r-1)}$, respectively. The law of continuation to the end of the table of probable duration of human life is very evident. The terms of the series which thus arises are as follows:

$$\text{For the 1st. year } \frac{a-b}{aR} S \times \frac{R-1}{m(r-1)},$$

$$\text{2d. } \frac{b-c}{aR^2} S \times \frac{R-1}{m(r-1)},$$

$$\text{3d. } \frac{c-d}{aR^3} S \times \frac{R-1}{m(r-1)},$$

$$\text{4th. } \frac{d-e}{aR^4} S \times \frac{R-1}{m(r-1)},$$

$$\text{5th. } \frac{e-f}{aR^5} S \times \frac{R-1}{m(r-1)}, \&c. \&c. \text{ to the}$$

end of the table; i. e. till one of the fractional factors becomes $= 0$.

6. Now, since the factor $\frac{R-1}{m(r-1)}$ (which, for the sake of brevity, I denote by F), is common to all the terms, as well as the factor S , if we can find any easy way of computing the sum of the finite series

$$\frac{a-b}{aR} + \frac{b-c}{aR^2} + \frac{c-d}{aR^3} + \frac{d-e}{aR^4} + \frac{e-f}{aR^5}, \&c. \text{ we}$$

shall readily obtain the present value of all the chances which the executor has to receive the sum S . But the sum of this series is evidently equal to the difference of the sums of these two series,

$$\begin{aligned} \text{viz. } & \frac{a}{aR} + \frac{b}{aR^2} + \frac{c}{aR^3} + \frac{d}{aR^4} + \frac{e}{aR^5}, \&c. \\ \text{and } & \frac{b}{aR} + \frac{c}{aR^2} + \frac{d}{aR^3} + \frac{e}{aR^4} + \frac{f}{aR^5}, \&c. \end{aligned} \left. \vphantom{\begin{aligned} \text{viz. } & \frac{a}{aR} + \frac{b}{aR^2} + \frac{c}{aR^3} + \frac{d}{aR^4} + \frac{e}{aR^5}, \&c. \\ \text{and } & \frac{b}{aR} + \frac{c}{aR^2} + \frac{d}{aR^3} + \frac{e}{aR^4} + \frac{f}{aR^5}, \&c. \end{aligned}} \right\} \begin{array}{l} \text{till the nu-} \\ \text{merator be-} \\ \text{comes } = 0. \end{array}$$

The first of these series is manifestly $=$

$$\frac{1}{R} \times \left(1 + \frac{b}{aR} + \frac{c}{aR^2} + \frac{d}{aR^3} + \frac{e}{aR^4}, \&c. \right) = \frac{1}{R} \times (1 + A)$$

$(1 + A)$, the letter A being put for the value of an annuity of £1 on the life of the person proposed in the question. And since the second series is obviously $= A$, we have their difference $= \frac{1+A}{R} - A$, and thence P , the value sought, $= FS \times \left(\frac{1+A}{R} - A \right) = FS \times \frac{1 - (R-1)A}{R} = FS \times \frac{1 - qA}{R}$.

7. Here it is worthy of remark, that the theorem now in general use is $P = S \times \left(\frac{1+A}{R} - A \right)$. Whatever ratio therefore, F shall be found to bear to 1, such will be the ratio of the true value of the insurance to that which is given by the rule now in use.

8. But the value of $F \left(= \frac{R-1}{m(r-1)} \right)$ is easily obtained thus: since $r^m = R = 1 + q$, we have $r = R^{\frac{1}{m}} = (1+q)^{\frac{1}{m}} = 1 + \frac{1}{m}q + \frac{1-m}{m \cdot 2m}q^2 + \frac{1-m}{m \cdot 2m} \times \frac{1-2m}{3m}q^3 + \frac{1-m}{m \cdot 2m} \times \frac{1-2m}{3m} \times \frac{1-3m}{4m}q^4$, &c. and $m(r-1) = q + \frac{1-m}{2m}q^2 + \frac{1-m}{2m} \times \frac{1-2m}{3m}q^3 + \frac{1-m}{2m} \times \frac{1-2m}{3m} \times \frac{1-3m}{4m}q^4$, &c. which expression, as m is greater than 31000000, is so nearly $= q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4$, &c. that the difference, in this case, is inconsiderable. And this series is known to be $=$ the hyperbolic logarithm* of $1 + q$, or R , which logarithm may be denoted by L . We therefore now have $F = \frac{R-1}{L}$, by which expression, and a table of hyperbolic logarithms, its numerical value may quickly be obtained, when the value of R is given. Its value may easily be found also by series, as follows: $F = \frac{R-1}{L} = q \div (q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4, \&c.) = 1 + \frac{1}{2}q - \frac{1}{12}q^2 + \frac{1}{24}q^3, \&c.$ which series, as q is a small quantity, is evidently greater than 1. It will indeed be nearly equal to, but always somewhat greater than, $\sqrt{1+q} = \sqrt{R}$.

9. The general solution of the problem being now completed, an example or two, by way of illustration, may be proper.

Example

* See the *Scriptores Logarithmici*, vol. i. pa. 244 and 245.

Example 1. Let it be required to find the numerical value of F , when $R = 1.03$, which is the common rate of interest allowed at the offices of insurance on lives.

Here $R - 1 = 0.03$; and the hyp. log. of $R = 0.029558 = L$; and $(R - 1) \div L$ is $= 1.014926$, the value of F required.

The same value of F may be found by the series given for that purpose in the preceding article, without the use of hyperbolic logarithms, thus :

$$\begin{aligned} 1 + \frac{1}{2}q &= 1.0150000, \\ + \frac{1}{24}q^2 &= 0.0000011, \\ \text{The sum} &= 1.0150011, \\ - \frac{1}{12}q^3 &= -0.0000750, \end{aligned}$$

The difference $= 1.0149261$, the value of F required.

If we now divide 1 by 0.014926 , the quotient will be 66.997 , or 67 very nearly; which shows that F exceeds 1 by somewhat more than a 67th part of itself. The loss to the offices, therefore, by the erroneous rule now in use, is somewhat more than £1 upon every £67 which they receive; it is indeed full £149 5s. upon every £10000.

If the rate of interest allowed were greater than £3 per cent. the error of the rule would also be greater than that which has been now computed.

Example 2. Let it be required to find the present value of £1000 to be received at the death of a person now 50 years of age, and in good health, taking the probable duration of human life to be such as is shown by the Northampton table, and allowing 3 per cent. interest of money.

Here $A = 12.436$, $= R = 1.03$, and $S = 1000$; from which we have $qA = 0.37308$, $1 - qA = 0.62692$, $(1 - qA) \div R = 0.60866$, and $S \times (1 - qA) \div R = £608.66$, the answer required, by the rule now in use.

The value of the annuity here used, viz. 12.436 , is taken from Dr. Price's *Observations on Reversionary Payments*, Edit. 6th, vol. ii. p. 315, where the numbers are continued to no more than three places of decimals, in consequence of which the last figure in the above calculation could not be depended upon, if it were not found by a different computation by the same erroneous rule. Now if we multiply the number 608.66 by 1.014926 , we shall have the true answer (viz. $FS \times (1 - qA) \div R$) $= £617.744$, which exceeds the erroneous answer by £9.084, or £9 1s. 8d.

REMARKS.

REMARKS.

1. This error deserves the attention of the directors of offices for insuring sums of money on lives. It is indeed said, that the proprietors of an office, in which a table of rates, grounded on the Northampton table and computed by the aforesaid erroneous theorem, has been used for the space of twenty years, have made considerable gain by it. This may be true: and it may be accounted for by observing, that the expectation of human life is, in general, somewhat longer than is shown by the Northampton table; and that the interest of money, during that period, has been more than 3 *per cent*, which is the rate allowed to those who make insurances at that office. *But still their gain is not so great as they imagine.*

The present competition of offices of this kind, for the favour of the public, may possibly induce some of them to offer lower rates of insurance, grounded on a table which shows the duration and gradual waste of human life to be such as it generally is in this Island; and should such tables be constructed by the theorem which has long been in use, that is, on the erroneous principle which I have now pointed out, the consequence must, in time, be ruinous to the proprietors, and to a multitude of widows and orphans, who must be deprived of that support, in the hour of distress, which was purchased for them by their husbands and fathers. Those proprietors therefore, who pay a due attention to justice, and to their own permanent interest, will not lower the present rates of insurance, without consulting some very able mathematician, who also has had much experience in calculations of this kind.

2. I must not omit to mention, that the same erroneous principle, which is above pointed out, has been used in constructing the tables of rates for insurance of sums of money when more than one life is concerned; in which case the effect of it varies with the number of lives.

3. I ought also to add, that I have neither private interest to serve, nor personal enmity to gratify, on this occasion. The course of my studies led me to an acquaintance with the doctrine of annuities and insurance on lives in my youth; and the many applications which have been made to me, of late years, by friends and acquaintance, for my opinion of the valuation of insurances on lives, have induced me to examine the principles on which the tables that have long been used for that purpose are constructed; and the examination has led me to the discovery of the error which is above demonstrated; a discovery which I take to be of no small importance to many individuals, if not to the public at large.

PHILALETES CANTABRIGIENSIS.

ARTICLE

ARTICLE XII.

An investigation of Theorems for finding the sums of certain Infinite Series, &c. By Mr. JAMES CUNLIFFE.

It has been shewn at art. 5, page 16, that

$$\frac{s}{1 - 2cx + x^2} = \sin. z + x \sin. 2z + x^2 \sin. 3z + x^3 \sin. 4z + \&c.$$

where s and c denote the sine and cosine of the circular arc z , radius 1.

1. Multiplying the preceding expression by $\dot{x}$, and taking the fluents, $\int \frac{s\dot{x}}{1 - 2cx + x^2} = R = x \sin. z + \frac{1}{2}x^2 \sin. 2z + \frac{1}{3}x^3 \sin. 3z + \frac{1}{4}x^4 \sin. 4z + \&c.$ where R denotes the length of a circular arc, radius 1 and tangent $\frac{sx}{1 - cx}$, or sine $\frac{sx}{\sqrt{(1 - 2cx + x^2)}}$.

2. In the place of c in the last expression write $-c$, and it becomes $\int \frac{s\dot{x}}{1 + 2cx + x^2} = S = x \sin. z - \frac{1}{2}x^2 \sin. 2z + \frac{1}{3}x^3 \sin. 3z - \frac{1}{4}x^4 \sin. 4z + \&c.$ where S denotes the length of a circular arc, radius 1 and tangent $\frac{sx}{1 + cx}$, or sine $\frac{sx}{\sqrt{(1 + 2cx + x^2)}}$.

3. Half the sum of the expressions 1 and 2 is $\frac{1}{2}(R + S) = x \sin. z + \frac{1}{3}x^3 \sin. 3z + \frac{1}{5}x^5 \sin. 5z + \frac{1}{7}x^7 \sin. 7z + \&c.$ where R and S are as before stated; or $R + S$ denotes the length of a circular arc, radius 1 and tangent $\frac{2sx}{1 - x^2}$, or sine $\frac{2sx}{\sqrt{\{1 + 2x^2 \times (1 - 2c^2) + x^4\}}} = \frac{2sx}{\sqrt{\{1 + 2x^2 \times (2s^2 - 1) + x^4\}}}$.

4. Again half the difference of the expressions 1 and 2 is $\frac{1}{2}(R - S) = \frac{1}{2}x^2 \sin. 2z + \frac{1}{4}x^4 \sin. 4z + \frac{1}{6}x^6 \sin. 6z + \&c.$ where $R - S$ denotes the length of a circular arc, radius 1 and

and tangent $\frac{2scx^3}{1+x^2 \times (s^2-c^2)}$, or sine

$$\frac{2scx^3}{\sqrt{\{1+2x^2 \times (1-2c^2)+x^4\}}} = \frac{2scx^3}{\sqrt{\{1+2x^2 \times (2s^2-1)+x^4\}}}.$$

Examples to the formula No. 3.

Example 1. Let z denote an arc of 45° : then $\sin. z = \sqrt{\frac{1}{2}}$; $\sin. 3z = \sqrt{\frac{1}{2}}$; $\sin. 5z = -\sqrt{\frac{1}{2}}$; $\sin. 7z = -\sqrt{\frac{1}{2}}$ &c. which being written in the formula No. 3, give $\frac{1}{2}(R+S) = \sqrt{\frac{1}{2}} \times (x + \frac{1}{2}x^3 - \frac{1}{2}x^5 - \frac{1}{2}x^7 + \frac{1}{2}x^9 + \frac{1}{12}x^{11} - \frac{1}{12}x^{13} - \frac{1}{12}x^{15} + \&c.)$ and dividing by $\sqrt{\frac{1}{2}}$, $\frac{1}{2}(R+S)\sqrt{2} = x + \frac{1}{2}x^3 - \frac{1}{2}x^5 - \frac{1}{2}x^7 + \frac{1}{2}x^9 + \frac{1}{12}x^{11} - \frac{1}{12}x^{13} - \frac{1}{12}x^{15} + \&c.$ where $R+S$ denotes the length of a circular arc, radius 1 and tangent $\frac{x\sqrt{2}}{1-x^2}$, or sine $\frac{x\sqrt{2}}{\sqrt{(1+x^2)}}$.

Now the series $x + \frac{1}{2}x^3 - \frac{1}{2}x^5 - \frac{1}{2}x^7 + \&c.$ is manifestly $= \int \frac{x}{1+x^4} + \int \frac{x^3 x}{1+x^4}$; therefore the sum of these two fluents is equal to the length of a circular arc, radius 1 and tangent $\frac{x\sqrt{2}}{1-x^2}$, or sine $\frac{x\sqrt{2}}{\sqrt{(1+x^2)}}$.

Take $x=1$, and then $\frac{1}{2}(R+S)\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{12} - \frac{1}{12} - \frac{1}{12} + \&c. = \frac{4}{1.3} - \frac{12}{5.7} + \frac{20}{9.11} - \frac{28}{13.15} + \&c.$

where $R+S$ in the present case denotes the circular arc, radius 1 and sine 1, or the quadrantal arc.

Example 2. Let z denote an arc of 60° ; then $\sin. z = \frac{1}{2}\sqrt{3}$; $\sin. 3z = 0$; $\sin. 5z = -\frac{1}{2}\sqrt{3}$; $\sin. 7z = \frac{1}{2}\sqrt{3}$; $\sin. 9z = 0$; $\sin. 11z = -\frac{1}{2}\sqrt{3}$; $\sin. 13z = \frac{1}{2}\sqrt{3}$; $\sin. 15z = -\frac{1}{2}\sqrt{3}$, &c. and these values being written in the formula No. 3, give $\frac{1}{2}(R+S) = \frac{1}{2}\sqrt{3} \times (x - \frac{1}{2}x^3 + \frac{1}{2}x^7 - \frac{1}{12}x^{11} + \frac{1}{12}x^{15} - \frac{1}{12}x^{19} + \&c.)$ and dividing by $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}(R+S) = x - \frac{1}{2}x^3 + \frac{1}{2}x^7 - \frac{1}{12}x^{11} + \frac{1}{12}x^{15} - \frac{1}{12}x^{19} + \&c.$

Now the series $x - \frac{1}{2}x^3 + \frac{1}{2}x^7 - \frac{1}{12}x^{11} + \frac{1}{12}x^{15} - \frac{1}{12}x^{19} + \&c. = \int \frac{x}{1-x^4} - \int \frac{x^4 x}{1-x^4}$; therefore the difference of

the

the fluents $\int \frac{x^2}{1-x^6}$ and $\int \frac{x^4 x^2}{1-x^6}$ is equal to $\frac{1}{\sqrt{3}} \times (R+S)$ where $R+S$ denotes the length of a circular arc, radius 1 and tangent $\frac{x\sqrt{3}}{1-x^2}$, or fine $\frac{x\sqrt{3}}{\sqrt{(1+x^2+x^4)}}$.

Take $x=1$; then $\frac{1}{\sqrt{3}} \times (R+S) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c. = \frac{4}{1.5} - \frac{4}{7.11} + \frac{4}{13.17} - \frac{4}{19.23} + \&c.$ and dividing by 4, $\frac{1}{4\sqrt{3}} \times (R+S) = \frac{1}{1.5} - \frac{1}{7.11} + \frac{1}{13.17} - \frac{1}{19.23} + \&c.$ in which circumstance $R+S$ denotes the quadrantal arc of a circle radius 1.

Example 3. Let z denote an arc of 30° ; then $\sin. z = \frac{1}{2}$; $\sin. 3z = 1$; $\sin. 5z = \frac{1}{2}$; $\sin. 7z = -\frac{1}{2}$; $\sin. 9z = -1$; $\sin. 11z = -\frac{1}{2}$ &c. and these being written in the formula No. 2. give $\frac{1}{2}(R+S) = \frac{1}{2} \times (x + \frac{3}{2}x^3 + \frac{1}{2}x^5 - \frac{1}{2}x^7 - \frac{3}{2}x^9 - \frac{1}{2}x^{11} + \&c.)$, or $R+S = x + \frac{3}{2}x^3 + \frac{1}{2}x^5 - \frac{1}{2}x^7 - \frac{3}{2}x^9 - \frac{1}{2}x^{11} + \&c.$ where $R+S$ denotes the length of a circular arc, radius 1 and tangent $\frac{x^2}{1-x^2}$.

Now the series $x + \frac{3}{2}x^3 + \frac{1}{2}x^5 - \frac{1}{2}x^7 - \frac{3}{2}x^9 - \frac{1}{2}x^{11} + \&c.$ is pretty evidently $= \int \frac{x^2}{1+x^6} + \int \frac{2x^2 x^2}{1+x^6} + \int \frac{x^4 x^2}{1+x^6}$; therefore the sum of these three fluents is equal to the length of a circular arc, radius 1 and tangent $\frac{x}{1-x^2}$.

Examples to the formula No. 4.

Example 1. Let z denote an arc of 45° ; then $\sin. 2z = 1$; $\sin. 4z = 0$; $\sin. 6z = -1$; $\sin. 8z = 0$; $\sin. 10z = 1$; $\sin. 12z = 0$; $\sin. 14z = -1$ &c. and these being written in the formula No. 4. give $\frac{1}{2}(R-S) = \frac{1}{2}x^2 - \frac{1}{2}x^6 + \frac{1}{2}x^{10} - \frac{1}{2}x^{14} + \&c.$ or $R-S = x^2 - \frac{1}{2}x^6 + \frac{1}{2}x^{10} - \frac{1}{2}x^{14} + \&c.$ where $R-S$ denotes the length of a circular arc, radius 1 and tangent x^2 , and this is also known from more simple principles.

Example

Example 2. Let z denote an arc of 60° : then $\sin 2z = \frac{1}{2}\sqrt{3}$; $\sin 4z = -\frac{1}{2}\sqrt{3}$; $\sin 6z = 0$; $\sin 8z = \frac{1}{2}\sqrt{3}$; $\sin 10z = -\frac{1}{2}\sqrt{3}$; $\sin 12z = 0$ &c. and these being written in the formula No. 4, give $\frac{1}{2}(R - S) = \frac{1}{2}\sqrt{3} \times (\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{8}x^6 - \frac{1}{16}x^8 + \&c.)$, and dividing by $\frac{1}{2}\sqrt{3}$ gives $\frac{1}{\sqrt{3}} \times (R - S) = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{8}x^6 - \frac{1}{16}x^8 + \&c.$ where $R - S$ denotes the length of a circular arc radius 1, and tangent $\frac{\frac{1}{2}x^2\sqrt{3}}{1 - \frac{1}{2}x^2} = \frac{x^2\sqrt{3}}{2 - x^2}$. Now $\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{8}x^6 - \frac{1}{16}x^8 + \&c.$ is pretty manifestly equal to $\int \frac{x\dot{x}}{1 - x^2} - \int \frac{x^3\dot{x}}{1 - x^2}$; therefore, $\frac{1}{\sqrt{3}} \times$ into the circular arc radius 1, and tangent $\frac{x^2\sqrt{3}}{2 - x^2}$ is equal to the difference of the fluents $\int \frac{x\dot{x}}{1 - x^2}$ and $\int \frac{x^3\dot{x}}{1 - x^2}$.

Take $x = 1$, then $\frac{1}{\sqrt{3}} \times (R - S) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \&c.$
 $= \frac{2}{2.4} + \frac{2}{8.10} + \frac{2}{14.16} \&c.$ and multiplying by 2, $\frac{2}{\sqrt{3}} \times (R - S)$
 $= \frac{1}{1.2} + \frac{1}{4.5} + \frac{1}{7.8} \&c.$ in which circumstance $R - S$ denotes a circular arc radius 1, and tangent $\frac{1}{\sqrt{3}}$: or $R - S$ in the present case is an arc of 30° .

Example 3. Let z denote an arc of 30° ; then $\sin 2z = \frac{1}{2}\sqrt{3}$; $\sin 4z = \frac{1}{2}\sqrt{3}$; $\sin 6z = 0$; $\sin 8z = -\frac{1}{2}\sqrt{3}$; $\sin 10z = -\frac{1}{2}\sqrt{3}$; $\sin 12z = 0$, &c. and these being written in the formula No. 4, give $\frac{1}{2}(R - S) = \frac{1}{2}\sqrt{3} \times (\frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6 - \frac{1}{16}x^8 + \&c.)$, and dividing by $\frac{1}{2}\sqrt{3}$, $\frac{1}{\sqrt{3}} \times (R - S) = \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6 - \frac{1}{16}x^8 + \&c.$ where $R - S$ denotes the length of a circular arc, radius 1, and tangent $\frac{\frac{1}{2}x^2\sqrt{3}}{1 - \frac{1}{2}x^2} = \frac{x^2\sqrt{3}}{2 - x^2}$.

Now $\frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6 - \frac{1}{16}x^8 + \&c. = \int \frac{x\dot{x}}{1 + x^2} + \int \frac{x^3\dot{x}}{1 + x^2}$; therefore the sum of the fluents $\int \frac{x\dot{x}}{1 + x^2} + \int \frac{x^3\dot{x}}{1 + x^2}$,

is equal to $\frac{1}{3\sqrt{3}} \times$ circular arc radius 1, and tangent $\frac{x^2\sqrt{3}}{2-x^2}$.

Take $x = 1$, then $\frac{1}{\sqrt{3}} \times (R - S) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \frac{1}{64} - \frac{1}{128} - \frac{1}{256} + \&c. = \frac{6}{2.4} - \frac{18}{8.10} + \frac{30}{14.16} - \frac{42}{20.22} + \&c.$
and multiplying by $\frac{2}{3\sqrt{3}}$, $\frac{2}{3\sqrt{3}} \times (R - S) = \frac{4}{2.4} - \frac{12}{8.10} + \frac{20}{14.16} - \frac{28}{20.22} + \&c. = \frac{1}{1.2} - \frac{3}{4.5} + \frac{5}{7.8} - \frac{7}{10.11} + \&c.$ where $R - S$ denotes a circular arc, radius 1, and tangent $\sqrt{3}$, or an arc of 60° .

ARTICLE XIII.

On the attraction of an infinite solid elliptic cylinder. By Mr. Thomas Knight, of Papcastle, near Cockermouth.

To the Editor of the Mathematical Repository.

SIR,

I send the following Problem for your Repository, because it contains a fluent, the finding of which is somewhat laborious, and as it will introduce some remarks I have to make on a passage of the *Mécanique Céleste* of La Place.

LEMMA. Let $\dot{M}$ be the element of a solid body, f its distance from an attracted point, then, the force being inversely as the square of the distance, $\dot{M} \div f^2$ is the attraction of that element on the point. By resolving this, we shall easily see that the attraction of the point by $\dot{M}$ in the direction of any other line u is $\frac{\dot{M}}{f^2} \times \left(\frac{f}{u}\right)$; whence the attraction of the whole solid in the direction u is $\int \frac{\dot{M}}{f^2} \times \left(\frac{f}{u}\right) =$ (if we put $\int \frac{\dot{M}}{f} = V$) $-\left(\frac{V}{u}\right).$

PROBLEM

PROBLEM.

Let XX, (fig. 19, pl. C,) be an infinite solid elliptic cylinder, p a point lying in the prolongation of the transverse diameter of the generating ellipsis. It is required to find the attraction the solid exercises on the point?

Let rq be any line in the solid, parallel to its axis; r the point where it meets a plane perpendicular to the axis, and passing through p ; mr a line perpendicular to Ap , or to the transverse axis of the generating ellipsis, A being its centre.

Put $pA = u$, $rq = t$, $Am = x$, $mr = y$, $pq = f = \sqrt{\{(u-x)^2 + y^2 + t^2\}}$. Suppose p to be attracted by an ele-

ment placed at q , we have, (because $\dot{M} = i\dot{y}\dot{x}$) $\int \frac{\dot{M}}{f} = V =$

$$\iiint \frac{i\dot{y}\dot{x}}{\sqrt{\{(u-x)^2 + y^2 + t^2\}}}; \text{ whence the attraction of the}$$

solid in the direction u , on the particle $= -(\frac{\dot{V}}{u})$, by the lemma,

$$= \iiint \frac{i\dot{y}\dot{x}(u-x)}{\{\{(u-x)^2 + y^2 + t^2\}^{\frac{3}{2}}\}}.$$

Let the equation of the generating ellipsis be $\tau^2 y^2 = k^2 - x^2$. The fluent relative to t must be taken from $t = -\infty$ to $t = \infty$:

the fluent relative to y from $y = \frac{-\sqrt{(k^2 - x^2)}}{\tau}$ to $y =$

$\frac{\sqrt{(k^2 - x^2)}}{\tau}$; and lastly that relative to x from $x = -k$ to x

$= k$. We have then successively,

$$-(\frac{\dot{V}}{u}) = \iiint \frac{i\dot{y}\dot{x}(u-x)}{\{(u-x)^2 + y^2 + t^2\}^{\frac{3}{2}}} = \iint \frac{\dot{y}\dot{x}(u-x)}{\{(u-x)^2 + y^2\}^{\frac{3}{2}}} = \int \dot{x} \times \text{ang. tang.} \frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)}.$$

The last fluent is the only one of any difficulty. Because $\int u\dot{v} =$

$uv - \int v\dot{u}$, we have first $\int \dot{x} \times \text{ang. tang.} \frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} =$

$$x \times \text{ang. tang.} \frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} - \int x \frac{\left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)}\right)' }{1 + \left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)}\right)^2}, \text{ the for-}$$

mer part of which vanishes when taken from $x = -k$ to $x = k$. There remains then to find

$$\int x \frac{\left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} \right)^{\frac{1}{2}}}{1 + \left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} \right)^2} = \int \left\{ + \frac{\tau x \sqrt{(k^2 - x^2)}}{\tau^2 (u-x)^2 + k^2 - x^2} - \frac{\tau(u-x) x^2 \dot{x}}{(\tau^2 (u-x)^2 + k^2 - x^2) \sqrt{(k^2 - x^2)}} \right\}$$

In order to make this rational, put $\frac{-(k+x)}{x-k} = z^2$, whence

$x = \frac{kz^2 - k}{z^2 + 1}$, $\sqrt{(k^2 - x^2)} = \frac{2kz}{z^2 + 1}$, $\dot{x} = \frac{4kz\dot{z}}{(z^2 + 1)^2}$. By substituting these values and making proper reductions, we find

$$\int x \frac{\left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} \right)^{\frac{1}{2}}}{1 + \left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} \right)^2} = 2d \times \frac{z^4 - z^2 - az^2 + a}{(z^2 + 1)(z^2 + g)(z^2 + f)} \times \dot{z},$$

where $g = \frac{b}{2} - \sqrt{\left(\frac{b^2}{4} - a^2\right)}$, $f = \frac{b}{2} + \sqrt{\left(\frac{b^2}{4} - a^2\right)}$,

$a = \frac{u+k}{u-k}$, $d = \frac{k^2}{\tau(u-k)}$, $b = \frac{2(u+k)}{u-k} + \frac{4k^2}{\tau^2(u-k)^2}$; it

is plain that $\frac{b^2}{4} - a^2$ is positive, as are also f and g . By resolving the above fraction, we have

$$\int x \frac{\left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} \right)^{\frac{1}{2}}}{1 + \left(\frac{\sqrt{(k^2 - x^2)}}{\tau(u-x)} \right)^2} = 2d \left\{ \frac{(2+2a)\dot{z}}{(f-1)(g-1)(z^2+1)} + \frac{(g+1)(g+a)\dot{z}}{(1-g)(f-g)(z^2+g)} + \frac{(f+1)(f+a)\dot{z}}{(1-f)(g-f)(z^2+f)} \right\}$$

whose fluent

$$= 2d \left\{ \frac{2+2a}{(f-1)(g-1)} \times \text{Arc. tan } z + \frac{(g+1)(g+a)}{(1-g)(f-g)\sqrt{g}} \times \text{Arc. tan } \frac{z}{\sqrt{g}} + \frac{(f+1)(f+a)}{(1-f)(g-f)\sqrt{f}} \times \text{Arc. tan } \frac{z}{\sqrt{f}} \right\},$$

which taken from $x = -k$, to $x = k$, gives

$$2d \left\{ \frac{2+2a}{(f-1)(g-1)} + \frac{(g+1)(g+a)}{(1-g)(f-g)\sqrt{g}} + \frac{(f+1)(f+a)}{(1-f)(g-f)\sqrt{f}} \right\} \pi,$$

π being half the circumference when the radius is unity. To reduce this to a simpler form, we have

$$f = \frac{u+k}{u-k} + \frac{2k^2}{\tau^2(u-k)^2} + \frac{2k}{\tau(u-k)^2} \sqrt{(u^2 - \frac{k^2(\tau^2-1)}{\tau^2})},$$

$$g = \frac{u+k}{u-k} + \frac{2k^2}{\tau^2(u-k)^2} - \frac{2k}{\tau(u-k)^2} \sqrt{(u^2 - \frac{k^2(\tau^2-1)}{\tau^2})},$$

$$fg =$$

$$f/g = \frac{(u+k)^2}{(u-k)^2}, f-g = \frac{4k}{\tau(u-k)^2} \sqrt{u^2 - \frac{k^2(\tau^2-1)}{\tau^2}},$$

$$\sqrt{f} = \frac{k+\tau\sqrt{u^2 - \frac{k^2(\tau^2-1)}{\tau^2}}}{\tau(u-k)}, \sqrt{g} = \frac{-k+\tau\sqrt{u^2 - \frac{k^2(\tau^2-1)}{\tau^2}}}{\tau(u-k)}$$

and by substituting these values, we find at last, the attraction of the solid

$$= - \left(\frac{\dot{V}}{u} \right) = \frac{4\pi\tau}{\tau^2-1} \left\{ u - \sqrt{u^2 - \frac{k^2(\tau^2-1)}{\tau^2}} \right\}.$$

SCHOLIUM. In the investigation by Mr. La Place, of the figure of Saturn's Ring, *Méc.Cél.* Liv. 3, Chap. 6, it appears to me, that his conclusions are rendered invalid by an error which I shall point out. He has rightly shewn, that the partial differential equation $\left(\frac{ddV}{du^2} \right) + \left(\frac{ddV}{dz^2} \right) = 0$, and consequently, its integral $V = \phi(u+z\sqrt{-1}) + \psi(u-z\sqrt{-1})$, which belong to an infinite cylinder, are also nearly true relative to the ring, if u and z are very small with respect to a , (see his work). But it by no means follows, that the *form* of the function in the one case, has the least similitude, or its value any approximation to that in the other. Yet $\int \frac{dt}{\sqrt{\{(u-x)^2+y^2+z^2\}}}$

belongs accurately to an infinite cylinder, notwithstanding the circuitous manner in which he arrives at it, and by which it would appear that he does not mean it to belong to such a cylinder.

Relatively to a ring, whose radius a is infinite, and whose dimensions and likewise the distance u are small, with respect to a , he would have had the integral

$$\int \frac{(a+x) d\omega}{\sqrt{\{2a^2 + 2au + 2ax - (2a^2 + 2au) \cos \omega\}}},$$

from which to find the form of the function $\phi(u)$.

Let rs , (fig. 20, pl. C,) be a circular line, the breadth evanescent, mn a touching line in the same plane; surely a particle at d , will not be equally attracted by each of these lines. If that be the case, remove it to e , and the circumference will not attract it all, which is plainly absurd, and yet the evident conclusion from the author's reasoning.

ARTICLE

ARTICLE XIV.

Two Indeterminate Problems. By Mr. Cunliffe.

PROBLEM I.

To find two scalene triangles, such, that the lengths of the perpendiculars from the angles upon the opposite sides, and segments of the sides made by the said perpendiculars, may be expressed by rational numbers; and also, that the perimeters and areas of the two triangles may be equal, each to each respectively.

SOLUTION.

Let ACB and $A'CB'$, (fig. 21, 22, pl. C,) represent two triangles of the kind mentioned in the question. In order that the perpendiculars and segments of the sides made by the perpendiculars may be expressed by rational numbers, put $AC = m^2 + n^2$, $BC = p^2 + q^2$, $AD = m^2 - n^2$, $BD = p^2 - q^2$; then $CD = 2mn = 2pq$; also, put $A'C' = m'^2 + n'^2$, $B'C' = p'^2 + q'^2$, $A'D' = m'^2 - n'^2$, $B'D' = p'^2 - q'^2$, then $C'D' = 2m'n' = 2p'q'$.

Now by the question, the perimeters and areas of the two triangles are to be equal each to each, that is, the perimeter of the one triangle is to be equal to the perimeter of the other; and the areas are in like manner to be equal. But the area of a triangle is known to be equal to the rectangle under the perimeter and half the radius of the inscribed circle, therefore the radii of the inscribed circles of the two triangles, must be equal to each other. These things being noticed, we shall proceed with the notation of the triangle ACB , to obtain expressions for the perimeter, and the radius of the inscribed circle.

The perimeter of the triangle ACB , is $2(m^2 + p^2)$, and its area $\frac{1}{2}CD \times AB = mn(m^2 - n^2 + p^2 - q^2) = mn(m^2 - n^2 + p^2 - m^2 n^2 \div p^2) = (mn \div p^2)(p^2 m^2 - p^2 n^2 + p^4 - m^2 n^2)$, because $pq = mn$, or $q = mn \div p$; and dividing the area by half the perimeter, gives $(mn \div p^2)(p^2 - n^2)$, for the radius of the inscribed circle. And for the same reason, $2(m'^2 + p'^2)$, must express the perimeter of the triangle $A'CB'$, and $(m'n' \div p'^2)(p'^2 - n'^2)$ the radius of its inscribed circle. Now the perimeters of the two triangles are to be equal to each other, and also the radii of their inscribed circles, that is,

$$p^2 + m^2 = p'^2 + m'^2, \text{ and } (mn \div p^2)(p^2 - n^2) = (m'n' \div p'^2)(p'^2 - n'^2).$$

Take

Take $m' = m$, then $p' = p$, and the last of these equations, after being properly ordered, becomes $n(p^2 - n^2) = n'(p'^2 - n'^2)$; whence $p^2 = (n^2 - n'^2) \div (n - n') = n^2 + nn' + n'^2$. Put $p = an \div b - n'$;

then $p^2 = n^2 + nn' + n'^2 = (an \div b - n')^2 = a^2 n^2 \div b^2 - 2ann' \div b + n'^2$, which gives $n' = n(a^2 - b^2) \div (2ab + b^2)$: take $n = 2ab + b^2$, then $n' = a^2 - b^2$, $p = p' = an \div b - n' = a^2 + ab + b^2$, $q = mn \div p = m(2ab + b^2) \div (a^2 + ab + b^2)$, $q' = m'n' \div p' = m(a^2 - b^2) \div (a^2 + ab + b^2)$:

and therefore $AC = m^2 + n^2 = m^2 + (2ab + b^2)^2$,
 $BC = p^2 + q^2 = (a^2 + ab + b^2)^2 + m^2(2ab + b^2)^2 \div (a^2 + ab + b^2)^2$,
 $AB = m^2 - n^2 + p^2 - q^2 = m^2 - (2ab + b^2)^2 + (a^2 + ab + b^2)^2 - m^2(2ab + b^2)^2 \div (a^2 + ab + b^2)^2$, $CD = 2mn = 2m(2ab + b^2)$.

Also in the triangle $A'C'B'$, $A'C' = m'^2 + n'^2 = m^2 + (a^2 - b^2)^2$,
 $BC = p^2 + q^2 = (a^2 + ab + b^2)^2 + m^2(a^2 - b^2)^2 \div (a^2 + ab + b^2)^2$,

$A'B' = m'^2 - n'^2 + p'^2 - q'^2 = m^2 - (a^2 - b^2)^2 + (a^2 + ab + b^2)^2 - m^2(a^2 - b^2)^2 \div (a^2 + ab + b^2)^2$; $CD' = 2m'n' = 2m(a^2 - b^2)$, where a , b , and m may be taken at pleasure. But in order to have whole numbers for the sides of the triangle, it is evident that m must be some multiple of $a^2 + ab + b^2$.

Example. Take $a = 2$, and $b = 1$, then $n = 2ab + b^2 = 5$, $n' = a^2 - b^2 = 3$, $p = p' = a^2 + ab + b^2 = 7$, $q = mn \div p = 5m \div 7$, and $q' = m'n' \div p' = 3m \div 7$. In order to have whole numbers for the sides, in the present circumstance, it is necessary to take m some multiple of 7; take $m = 14$, then $q = 5m \div 7 = 10$, and $q' = 3m \div 7 = 6$; whence $AC = m^2 + n^2 = 196 + 25 = 221$, $BC = p^2 + q^2 = 49 + 100 = 149$, $AB = m^2 - n^2 + p^2 - q^2 = 120$, and $CD = 2mn = 140$. Also $A'C' = m^2 + n'^2 = 196 + 9 = 205$, $B'C' = p'^2 + q'^2 = 49 + 36 = 85$, $A'B' = m'^2 - n'^2 + p'^2 - q'^2 = 200$, and $CD = 2m'n' = 84$.

PROBLEM II.

To determine a polygon of any assigned number of sides, that may be inscribed in a circle, and such, that the sides of all the triangles, into which the polygon can possibly be divided, may all of them be expressed by rational numbers; and also, that the areas of the said triangles may every one of them be expressed by rational numbers.

SOLUTION

SOLUTION.

Let $ABCDE$, (fig. 23, pl. C,) be a figure of five sides that may be inscribed in a circle, and draw lines between every two angles thereof: Let BE , BD cut AC in the points M and N respectively, and draw BF perpendicular to AC .

Then by the well known property of the circle,
 $ME = AM \times MC \div BM$ and $ND = AN \times NC \div BN$.
 The triangles MCB , MEA , are similar, as also the triangles NBA , NCA ,

therefore $BM : AM :: BC : AE = AM \times BC \div BM$
 and $BN : NC :: AB : CD = NC \times AB \div BN$:

The triangles MAB , MEC ; NCB , NDB are also similar,
 therefore $BM : MC :: AB : CE = MC \times AB \div BM$,
 and $BN : AN :: BC : AD = AN \times BC \div BN$.

Again, by a known property of a trapezium inscribed in a circle $ED = (BE \times AD - BD \times AE) \div AB$.

From whence it is evident enough, when duly considered, that if the lines BA , BM , BN , and BC , together with AF , MF , CF , and NF are denoted by rational numbers, then the sides of all the triangles into which the figure $ABCDE$ can possibly be divided will be expressed by rational numbers. We shall now show that the areas of all the said triangles, will be expressed by rational numbers, when, besides the before mentioned conditions, the perpendicular BF is denoted by a rational number. The area of any triangle, inscribed in a circle, is equal to the continued product of the three sides thereof divided by twice the diameter of the circumscribing circle, as may be easily proved.

Now $BA \times BC \div BF =$ the diameter of the circle circumscribing the figure $ABCDE$, and this expression for the diameter will be rational when BA , BC , and BF are so; consequently the area of each of the triangles, into which the figure $ABCDE$ can be divided, being the continued product of three rational numbers divided by another rational number must be rational. We shall now proceed to apply what has been deduced to the obtaining of numerical values for the sides and diameter of the circumscribing circle of a five-sided figure answering the conditions of the problem.

Put $AB = m^2 + n^2$, $AF = m^2 - n^2$; $BM = m^2 + n^2$,
 $MF = m^2 - n^2$; $BN = m'^2 + n'^2$, $NF = m'^2 - n'^2$; $BC = m'^2 + n'^2$, and $CF = m'^2 - n'^2$; then $BF = 2mn = 2m'n'$
 $= 2m''n'' = 2m'''n'''$, or $\frac{1}{2}BF = mn = m'n' = m''n'' = m'''n'''$,
 which indicates that the number which denotes $\frac{1}{2}BF$ is to be resolved

resolved into two factors four different ways. Admitting fractional factors, any number whatever may be resolved into two factors in an infinite variety of ways, from whence we may infer, that the question admits of an infinite variety of answers. Perhaps the following assumption will give the least whole numbers possible for the sides of the five sided figure, and the diameter of the circumscribing circle.

Assume $mn = 24 \times 1 = 8 \times 3 = 6 \times 4 = 12 \times 2$; then we may evidently take

$m = 24, n = 1; m' = 8, n' = 3; m'' = 6, n'' = 4; m''' = 12, n''' = 2$; whence $AB = m^2 + n^2 = 577, AF = m^2 - n^2 = 575$; $BM = m'^2 + n'^2 = 73, MF = m'^2 - n'^2 = 55, BN = m''^2 + n''^2 = 52, NF = m''^2 - n''^2 = 20, BC = m'''^2 + n'''^2 = 148, CF = m'''^2 - n'''^2 = 140,$

$$ME = \frac{AM \times MC}{BM} = \frac{520 \times 195}{73}, BE = BM + ME = \frac{106729}{73},$$

$$ND = \frac{AN \times NC}{BN} = \frac{595 \times 120}{52}, BD = BN + ND = \frac{74104}{52},$$

$$AE = \frac{AM \times BC}{BM} = \frac{76960}{73}; CD = \frac{NC \times AB}{BN} = \frac{69240}{52},$$

$$CE = \frac{MC \times AB}{BM} = \frac{112515}{73}, AD = \frac{AN \times BC}{BN} = \frac{88060}{52},$$

$$ED = \frac{BE \times AD - BD \times AE}{AB} = \frac{6404700}{73 \times 52}, \text{ and the di-}$$

$$\text{ameter of the circumscribing circle } \frac{AB \times BC}{BF} = \frac{577 \times 148}{48}$$

$$= \frac{21349}{12}.$$

Integer values of the various parts of the figure may be readily obtained from those already found, viz. by reducing the said parts to a common denominator, and rejecting it. If what has been done is properly understood, there can remain no other difficulty in extending the question to a figure of a greater number of sides, but the unavoidable trouble of the calculation.

ARTICLE XV.

Two letters from Mr. Thomas Knight, of Papcastle, on the proportionality of the force to the velocity, and the composition of forces.

LETTER I.

On the proportionality of the force to the velocity.

To the Editor of the Mathematical Repository.

SIR,

I wish you could give the following paper in your next number.

Papcastle,
July 8th, 1808.

I am Sir, your most obedient Servant,

THOS. KNIGHT.

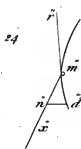
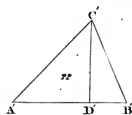
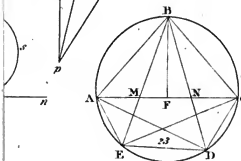
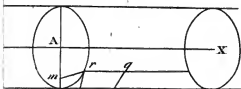
Is the proportionality of the force to the velocity, a necessary or contingent truth?

“ Si ce principe est de vérité nécessaire, nous avouerons que les preuves qu'on en a apportées jusqu'ici ne nous paroissent pas bien d'atteinte.” D'Alembert.

What is given for the composition of forces in most of our elementary books, is merely that of velocities, a matter which has no difficulty; but nothing can from thence be concluded respecting forces, till it be shewn, either by demonstration or experiment, what relation the force has to the velocity. There appears no reason why the one should be simply as the other. Might not a double force produce four times or eight times the motion? Indeed it implies (at first view) no contradiction to suppose the one any function whatever of the other. This being considered, many ingenious persons have investigated what shall be the force equivalent to two others, independently of any consideration of velocity. The two following propositions must be regarded as perfectly distinct, and have been so often proved, that I shall take them for granted.

1st. If there be impressed on a body a velocity AD (fig. 25, pl. D.) and another AB in a direction perpendicular to that of the former, the body will move in the diagonal AC, and with a velocity equal to that line.

2d. The effect of two forces AD, AB whose directions form
a right.



a right-angle, is equal to the effect of a force which acts in the direction of, and equals in quantity, the diagonal AC. *

I now come to the object of this paper; M. La Place affirms (see *Méc. cél.* Tome 1; page 15 to 18: also *Syst. du Monde*, p. 139, 140) that we cannot know the relation of the force to the velocity it generates, without having recourse to experience, and he shews by an ingenious analysis what may be the experiments proper to determine that relation: from them it appears that the force is as the velocity. This according to him is one of those facts—‘*That Mechanics borrows of experience;*’ and which reason alone could not have taught us;—‘*considering our ignorance of the nature of moving force.*’ I confess I am truly astonished that such should be his opinion; it seems to me on the contrary, that the proportionality of the force to the velocity is easily proved, that it is a *necessary truth*, and discovered without any aid from experience. For let AB , AD represent forces, whose directions form a right angle, giving separately the velocities AB , AD , or by the composition of velocities, giving the velocity AC in the direction AC . The resulting force (which by the composition of forces is the diagonal AY) is that which shall have the same effect as the composing ones: it must therefore give the velocity AC in the direction AC . Now if we suppose the force is not as the velocity, what is the consequence? That the ratio of AB to BC is different from that of AB to BY , consequently that AY and AC differ in direction: whence this notable conclusion, that the direction of the motion is different from that in which the action of the force is exercised; this being absurd, the proposition I proposed to demonstrate is evident.

I cannot see any error in the above proof, yet I offer it with diffidence for this reason, that so plain a matter, if true, could hardly have escaped the sagacity of such men as D’Alembert and La Place.†

LETTER II.

On the composition of forces.

To the Editor of the Mathematical Repository.

SIR,

I endeavoured, in a former letter, to shew that there was only one relation, *mathematically possible*, between force and velocity. The composition of forces had been investigated by many eminent men, and I had at that time nothing to add on the subject: a demonstration, however, has since occurred to me, which seems simpler than any hitherto given.

* See Letter II

† Note. I have taken nothing for granted in this paper. That the effect of a force in any direction will not be altered by a motion acting at right angles to that direction, I suppose no one will dispute. This is considered as an axiom by *Fontenair*, whose work was corrected by *La Place*.

In shewing what is the *quantity* of the resulting force, I follow M. La Place, *Méc. cél.*, but find the *direction*, which is the more difficult matter, by a much easier way, and without the assistance of the Integral Calculus.

Let x and y be two forces acting on a point, and forming a right-angle with each other, z the resulting force, and θ the angle formed by x and z . I shall suppose the first part demonstrated as by La Place: viz. that $x = zp(\theta)$, $y = zp(\frac{\pi}{2} - \theta)$, $x^2 + y^2 = z^2$ (α), ϕ being some unknown function. Because $\frac{x}{z}$ is a function of θ , it follows conversely that θ and consequently $\cos \theta$ are functions of $\frac{x}{z}$, or $\cos \theta = \psi(\frac{x}{z})$; in like manner $\cos(\frac{\pi}{2} - \theta) = \psi(\frac{y}{z})$. Let these functions be supposed expanded as follows

$$\left. \begin{aligned} \cos \theta &= 'A \frac{x}{z} + ''A \frac{x^3}{z^3} + ''''A \frac{x^5}{z^5} + \&c. \\ \cos(\frac{\pi}{2} - \theta) &= 'A \frac{y}{z} + ''A \frac{y^3}{z^3} + ''''A \frac{y^5}{z^5} + \&c. \end{aligned} \right\} \begin{array}{l} \text{It is plain} \\ \text{there can be} \\ \text{no even} \\ \text{powers.} \end{array}$$

By squaring each of these equations and adding them together, there arises $\cos^2 \theta + \cos^2(\frac{\pi}{2} - \theta) =$

$$\begin{aligned} 1 &= 'A \frac{x^2 + y^2}{z^2} + 2'A ''A \frac{x^4 + y^4}{z^4} + ''''A^2 \\ &\quad + 2'A ''''A \frac{x^6 + y^6}{z^6} + 2''A ''''''A \frac{x^8 + y^8}{z^8} + \&c. \dots (\beta). \end{aligned}$$

Now (α) gives $1 = \frac{x^2 + y^2}{z^2}$, and this equation can be no other than (β). For if it were possible that there could be two different relations between x , y and z , we might, by elimination, get an equation between z and x , that is, z would be a function of x alone, which is absurd. By comparing then these two equations, we find

$$\begin{aligned} 'A &= 1, 2'A ''A = 0, ''A^2 + 2'A ''''A = 0, 2''A ''''A + 2'A ''''''A \\ &= 0, ''''A^2 + 2''A ''''''A + 2'A ''''''''A = 0, \&c. \end{aligned}$$

and

and from these we get $''A = 0$, $'''A = 0$, $''''A = 0$, &c. therefore $\cos \theta = \frac{x}{z}$, which was the thing to be found.

SCHOLIUM. As Mr. La Place's works may not be in the hands of all your readers, I shall give his demonstration, of the first part of the composition of forces, alluded to in the above paper.

Let us suppose at first the forces x and y infinitely small and equal to dx and dy ; let us suppose next, that x becoming successively dx , $2dx$, $3dx$, &c. y becomes dy , $2dy$, $3dy$, &c. it is plain that the angle θ will always be the same, and that the resulting force z will become successively dz , $2dz$, $3dz$, &c. thus in the successive increments of x , y and z , the ratio of x to z will be constant, and may be expressed by a function of θ , which we will denote by $\phi(\theta)$; we shall have then $x = z \cdot \phi(\theta)$, in which equation we may change x into y , provided that we also change θ into $\frac{\pi}{2} - \theta$, π being the semi-circumference whose radius is unity. Now we may consider the force x as resulting from two forces x' and x'' , the first acting in the direction of z , and the second perpendicular to that direction. The force x which results from these two new forces, making the angle θ with the force x' and the angle $\frac{\pi}{2} - \theta$ with the force x'' , we shall have

$$x = x\phi(\theta) = \frac{x^2}{z}; \quad x' = x\phi\left(\frac{\pi}{2} - \theta\right) = \frac{xy}{z}; \quad \text{we may there-}$$

fore substitute these two forces in the place of x . We may in like manner put instead of y , two new forces y' and y'' , the first of which is equal to $\frac{y^2}{z}$ and in the direction of z , and the

other is $\frac{xy}{z}$ and perpendicular to z ; we have thus in place of the two forces x and y , the four following

$$\frac{x^2}{z}, \quad \frac{y^2}{z}, \quad \frac{xy}{z}, \quad \frac{xy}{z}.$$

The two last acting in opposite directions destroy one another; the others, acting in the same direction, added together will form the resulting force z ; we shall have then $x^2 + y^2 = z^2$.

Papcastle,
October 23, 1808.

I am Sir, Your Obedient Servant,
THOS. KNIGHT.

ARTICLE

ARTICLE XVI.

On the Composition of rotatory Motions. By Mr. KNIGHT.

To the Editor of the Mathematical Repository.

SIR,

In this paper I have demonstrated in an easy manner the propositions that have been given by *Frisi* and *La Grange*, concerning the composition of rotatory motions; to which I have added a still more general theorem.

Let a body have forces impressed on it, that would separately give it a movement of rotation, about the axis Cx , (fig. 28. pl. D.) with the angular velocity v ; and about Cy with the angular velocity v' . It is plain that a particle in the line C , will not be affected in its motion about Cx , by the other rotation about Cy . The same observation (*mutatis mutandis*) may be applied to a particle in the line Cx : but in any intermediate line in the plane yCx , as Cm , the motion round the one axis will impede that round the other; and if we can find a line in which all the particles are at rest, that will be the momentary axis of rotation composed of and equivalent to the two others.

Put $Cm = r$, angle $x Cm = \theta$, angle $y Cm = \theta'$.

A perpendicular from m on Cx will be $r \sin \theta$,

A perpendicular from m on Cy will be $r \sin \theta'$, and it is plain that m will be at rest if

$v' : v = r \sin \theta : r \sin \theta' = \sin \theta : \sin \theta' \dots\dots\dots (\alpha)$,
which determines the situation of the momentary axis.

To find the velocity round this, let us suppose ymx to be a circle drawn through m , the centre being C . Let ay be the velocity of the point y ; we have $ay = vr \sin xCy$. Now the

velocity about Cm at the distance r is plainly $ay \times \frac{r}{r \sin \theta} =$

$\frac{ay}{\sin \theta}$, consequently the velocity at distance unity, or the
angular vel. $= \frac{ay}{r \sin \theta} = v \times \frac{\sin xCy}{\sin \theta} \dots\dots\dots (\beta)$.

The theorems in equations (α) and (β) were first given by *Frisi*. If, besides the two rotations already mentioned, there had been given, at the same time, another round a third axis Cz , with an angular

angular velocity $= v'$; it is plain that the axis of rotation, arising from these three, would be in the plane zCm , as at Cm' , suppose it to decline from Cm by the angle θ' , and from Cz by θ'' ; we find, as we did (a) and (b),

$$v'' : v \times \frac{\sin xCy}{\sin \theta} = \sin \theta' : \sin \theta'' \dots\dots\dots (\gamma)$$

and angular velocity $= v \times \frac{\sin xCy}{\sin \theta} \times \frac{\sin zCm}{\sin \theta'} \dots\dots\dots (\delta).$

If, moreover, a rotation had been given round a fourth axis Cu , with an angular velocity $= v''$, we should have

$$v''' : v \times \frac{\sin xCy \cdot \sin zCm}{\sin \theta' \cdot \sin \theta''} = \sin \theta''' : \sin \theta'''' \dots\dots\dots (\epsilon),$$

θ''' and θ'''' being angles that the momentary axis declines from Cm' and Cu ; and the angular velocity $=$

$$v \times \frac{\sin xCy \cdot \sin zCm \cdot \sin uCm'}{\sin \theta' \cdot \sin \theta'' \cdot \sin \theta'''},$$

and thus we may proceed to any number of axes.

We will now consider the case in which the angles xCy , zCm , uCm' &c. are right, and their sines unity. We have $\sin \theta = \cos \theta'$, $\sin \theta'' = \cos \theta'''$ &c. here (a) becomes $v' : v = \cos \theta' :$

$$\sin \theta' = \sqrt{(1 - \sin^2 \theta')} : \sin \theta'; \text{ whence } \sin \theta' = \frac{v}{\sqrt{(v^2 + v'^2)}}.$$

$$\text{We get in like manner from } (\gamma), \sin \theta'' = \frac{\sqrt{(v^2 + v'^2)}}{\sqrt{(v^2 + v'^2 + v''^2)}},$$

$$\text{and from } (\epsilon) \sin \theta'''' = \frac{\sqrt{(v^2 + v'^2 + v''^2)}}{\sqrt{(v^2 + v'^2 + v''^2 + v'''^2)}}, \text{ and so}$$

on: and if the number of axes be n , and the velocities round those axes v , v' , v'' , $\dots\dots\dots v'''^{n-1}$, the angular velocity round the axis arising from the composition of them all will be

$$\sqrt{\{v^2 + v'^2 + v''^2 + \dots\dots\dots + (v'''^{n-1})^2\}}. \text{ Of this very}$$

general theorem the case considered by La Grange of three axes perpendicular to each other, is a particular case; in which ang.

$$\text{vel.} = \sqrt{(v^2 + v'^2 + v''^2)} \text{ and } \sin \theta' = \frac{v}{\sqrt{(v^2 + v'^2)}}, \sin \theta''' =$$

$$\frac{\sqrt{(v^2 + v'^2)}}{\sqrt{(v^2 + v'^2 + v''^2)}}, \text{ or, the angle } zmy \text{ being a right one, we easily find that } Cm' \text{ makes angles with } Cx, Cy, Cz \text{ whose cosines are}$$

$$\frac{v}{\sqrt{(v^2 + v'^2 + v''^2)}}, \frac{v'}{\sqrt{(v^2 + v'^2 + v''^2)}}, \frac{v''}{\sqrt{(v^2 + v'^2 + v''^2)}}.$$

In

In the general case I treated of above, the angles zmy , $zm'm$, &c. are any whatsoever, and if their magnitude be given, the situation of the composed axis is also given, by means of the values of $\sin \theta$, $\sin \theta''$, $\sin \theta'''$, &c.

I am Sir,

Papcastle,
Dec. 20, 1808.

Your obedient Servant,
THOS. KNIGHT.

ARTICLE XVII.

On the Expansion of certain Functions. By Mr. KNIGHT.

To the Editor of the Mathematical Repository.

SIR,

It seems not a little extraordinary that the following demonstrations should have escaped the notice of mathematicians, at least as far as I know.

To expand the exponential a^x .

Let $a^x = 1 + 'Ax + ''Ax^2 + ''''Ax^3 + \&c.$

Its fluxion gives

$$a^x = \frac{'A}{L.a} + \frac{''Ax}{L.a} + \frac{'''Ax^2}{L.a} + \frac{''''Ax^3}{L.a} + \&c.$$

Whence $'A = L.a$, $''A = \frac{(L.a)^2}{1.2}$, $'''A = \frac{(L.a)^3}{1.2.3}$, &c.

Let $a^{c+c'x+c''x^2} = A + 'Ax + ''Ax^2 + ''''Ax^3 + \&c.$

Then $a^{c+c'x+c''x^2} (c' + 2c''x) = \frac{'A}{L.a} + \frac{''Ax}{L.a} + \frac{'''Ax^2}{L.a} + \&c.$

Multiply the first by $c' + 2c''x$, and there arises

$$a^{c+c'x+c''x^2} (c' + 2c''x) = c'A + c' 'A x + c' ''A x^2 + \&c. \\ + 2c''A + 2c'' 'A$$

Whence by comparing the two last

$$\frac{'A}{L.a} = c'A, \frac{''A}{L.a} = c' 'A + 2c''A, \frac{'''A}{L.a} = c' ''A + 2c'' 'A, \&c.$$

But $A = a^c$, consequently all the other coefficients are known.

Let

Let $\sin x = x + \text{''}Ax^3 + \text{''''}Ax^5 + \&c.$ the fluxional coefficient of the second order gives

$$\sin x = -2.3 \text{''}Ax - 4.5 \text{''''}Ax^3 + \&c.$$

Whence $\text{''}A = -\frac{1}{2.3}$, $\text{''''}A = \frac{1}{2.3.4.5}$, &c.

In like manner is $\cos x$ found from its second fluxion; but both might have been found at once by means of their *first fluxions only*; in the manner following:

Let $\sin(w+x) = \sin w + \text{' }Ax + \text{''}Ax^2 + \text{'''}Ax^3 + \&c.$

and $\cos(w+x) = \cos w + \text{' }Bx + \text{''}Bx^2 + \text{'''}Bx^3 + \&c.$

Then $\cos(w+x) = \text{' }A + 2 \text{''}Ax + 3 \text{'''}Ax^2 + 4 \text{''''}Ax^3 + \&c.$

and $\sin(w+x) = -\text{' }B - 2 \text{''}Bx - 3 \text{'''}Bx^2 - 4 \text{''''}Bx^3 + \&c.$

From the two middle equations we find

$\text{' }A = \cos w$, $\text{' }B = 2 \text{''}A$, $\text{''}B = 3 \text{'''}A$, $\text{'''}B = 4 \text{''''}A$, &c.

from the extreme ones

$$\text{' }B = -\sin w, \text{''}B = -\frac{\text{' }A}{2}, \text{'''}B = -\frac{\text{''}A}{3} \&c.$$

Whence all the coefficients are known.

With equal ease are found the sine and cosine of a multinomial; I shall here confine myself to $a + a'x + a''x^2$. Let then

(α) $\sin(a + a'x + a''x^2) = A + \text{' }Ax + \text{''}Ax^2 + \text{'''}Ax^3 + \&c.$

(β) $\cos(a + a'x + a''x^2) = B + \text{' }Bx + \text{''}Bx^2 + \text{'''}Bx^3 + \&c.$

The fluxional coefficients of the first order are

(γ) $\cos(a + a'x + a''x^2)(a' + 2a''x) = \text{' }A + 2 \text{''}Ax + 3 \text{'''}Ax^2 + \&c.$

(δ) $\sin(a + a'x + a''x^2)(a' + 2a''x) = -\text{' }B - 2 \text{''}Bx - 3 \text{'''}Bx^2 - \&c.$

(α) and (β) multiplied by $a' + 2a''x$ produce

(ϵ) $\sin(a + a'x + a''x^2)(a' + 2a''x) = a' A + a'' A \left\{ \begin{matrix} x + a'' A \\ + 2a'' A \end{matrix} \right\} x^2 + \&c.$

(ζ) $\cos(a + a'x + a''x^2)(a' + 2a''x) = a' B + a'' B \left\{ \begin{matrix} x + a'' B \\ + 2a'' B \end{matrix} \right\} x^2 + \&c.$

Compare (δ) and (ζ) and there arises

$$a' A = -\text{' }B, \frac{a'' A}{2} + a'' A = -\text{''}B, \frac{a'' A}{3} + \frac{2a'' A}{3} = -\text{'''}B \&c.$$

From (γ) and (ζ) we find

$$a' B = \text{' }A, \frac{a'' B}{2} + a'' B = \text{''}A, \frac{a'' B}{3} + \frac{2a'' B}{3} = \text{'''}A, \&c.$$

But $A = \sin a$, $B = \cos a$, whence all the coefficients are known. Thus the sine and cosine of a multinomial are found at the same time, and by a simpler method than any I have seen.

In a manner not very different, may the elliptick sine and cosine be found at *one operation*, in terms of the elliptick arc; from whence the sine and cosine of the circle are deduced as a particular case, which, though of no great use, I shall add here on account of its novelty.

AM (fig. 29, pl. D.) is an ellipsis, AM a circle whose centre is that of the ellipsis, and its radius AC half the transverse diameter: let this = 1, and the semi-conjugate = m . PM is perpendicular to AC. Put AM = x , A'M = x' :

P'M is the elliptick sine of $x' = m$ sine of x , by conics; CP is the elliptick cosine of $x' = \cos x$. Let

elliptick $\sin x' = m \sin x = x' + \frac{1}{3} A x'^3 + \frac{1}{5} A^3 x'^5 + \frac{1}{7} A^5 x'^7 + \&c$

elliptick $\cos x' = \cos x = 1 + \frac{1}{2} B x'^2 + \frac{1}{24} B^3 x'^4 + \frac{1}{720} B^5 x'^6 + \&c$

whose fluxions are

$$m \cos x \dot{x} = \dot{x}' (1 + \frac{1}{3} A x'^2 + \frac{1}{5} A^3 x'^4 + \frac{1}{7} A^5 x'^6 + \&c.$$

$$\sin x \dot{x} = - \dot{x}' (\frac{1}{2} B x' + \frac{1}{24} B^3 x'^3 + \frac{1}{720} B^5 x'^5 + \&c.$$

By the nature of the circle $\sin^2 x + \cos^2 x = 1$; and in all

curves $\dot{AM}^2 = \dot{AP}^2 + \dot{PM}^2$ which is in the present case $(\dot{x}')^2 = \sin^2 x \dot{x}^2 + m \cos^2 x \dot{x}^2$.

If we substitute in these two equations the above expansions, there arises

$$1 = 1 + \frac{1}{m^2} \left\{ x'^2 + \frac{2}{m^2} A \right\} x'^4 + \frac{\frac{1}{m^2} A^2}{\frac{1}{m^2}} x'^6 + \&c. \text{ And} \\ + \frac{2}{m^2} B \left\{ \frac{1}{2} B x' + \frac{1}{24} B^3 x'^3 + \frac{1}{720} B^5 x'^5 + \&c \right\}$$

$$1 = 1 + \frac{2.3}{2.2} \frac{A}{B^2} \left\{ x'^2 + \frac{3.3}{2.5} \frac{A^2}{B^3} \right\} x'^4 + \frac{2.3.5}{2.2.4} \frac{A^3}{B^4} x'^6 + \&c. \\ + \frac{2.5}{2.2.4} \frac{A^2}{B^3} \left\{ \frac{1}{2} B x' + \frac{1}{24} B^3 x'^3 + \frac{1}{720} B^5 x'^5 + \&c \right\}$$

from which, by making separately equal to nothing the coefficients of the powers of x in each, we get two sets of equations, whence are derived the coefficients $A, A^3, \&c. B, B^3, \&c.$ and

$$\text{elliptick } \sin \text{ of } x' = x' - \frac{1}{2.3.m^4} x'^3 + \frac{13 - 12m^2}{2.3.4.5.m^6} x'^5 - \&c.$$

$$\text{elliptick } \cos \text{ of } x' = 1 - \frac{1}{2m^2} x'^2 + \frac{4 - 3m^2}{2.3.4.m^6} x'^4 - \&c.$$

By making $m = 1$, we get the common series for the sine and cosine

cosine of a circular arc : And, by reverting the series, the rectification of an elliptick arc is accomplished, which though it may not come out in a very useful form, is so far remarkable, that it has been (for the first time I believe) found without help from the integral calculus.

ARTICLE XVIII.

Two letters from Mr. KNIGHT, on the expansion of any Function of a Multinomial.

LETTER I.

To the Editor of the Mathematical Repository.

Sir,

The following paper contains an investigation of the expansion of any function of a multinomial, which has hitherto received no demonstration but that given by *M. Arbogast*, in his very learned work, '*Du calcul des derivations*'. As the method I use is quite different from his, I think it may not be uninteresting to the learned part of your readers.

Papcastle,

I am Sir, your's, &c.

Nov. 5, 1808.

THOS. KNIGHT.

It is well known to mathematicians that, if u be a function of $a', a'', a''', \&c.$ and expanded according to the powers and products of those quantities, the coefficient of $a'^{m'} a''^{m''} a'''^{m'''} \&c.$ will be

$$\left(\frac{m' + m'' + m''' +}{a'^{m'} a''^{m''} a'''^{m'''} \&c.} \right)$$

$1.2.3...m' \times 1.2.3...m'' \times 1.2.3...m''' \times \&c. \dots\dots (a)$

provided that in this expression we make $a', a'', a''', \&c.$ vanish after the indicated fluxional operation. I shall apply this to the following form, $f(A + c'a' + c''a'' + c'''a''' +)$, where $a', a'', a''', \&c.$ represent $x, x^2, x^3, \&c.$ being more manageable for the present. I shall denote by $f, f', f'', \&c.$ the successive fluxional coefficients of the function f .

The first thing to be observed is that the expression (a), when the quantities $a', a'', a''', \&c.$ have been made to vanish, will become

$$\frac{f(A) c^{m'} c^{m''} c^{m'''} \&c.}{1.2.3\dots m' \times 1.2.3\dots m'' \times 1.2.3\dots m''' \times \&c.} \dots\dots\dots (\beta).$$

Now let it be required to find the coefficient of any power (n) of x in the expansion of the above function.

Place in a horizontal line the successive powers of a' multiplied by one of the other quantities, so that the sum of the *strokes* shall be n . In a second line the powers of a' combined with two other quantities: and so on. Then proceed in the same manner with a' &c. Observe to let those quantities stand in the same perpendicular line, the sum of whose *exponents* forms the same number.

To illustrate this, let n be 6; by proceeding as above directed we form the following scheme;

$a^{''''''}$	$a' a^{''''}$	$a'^2 a^{''''}$	$a'^3 a^{''''}$	$a'^4 a^{''''}$	a'^6
		$a' a'' a^{''''}$	$a'^2 a^{''''}$		
	$a' a^{''''}$	$a^{''''}$			
	$a^{''''}$				
$f(A)$	$f(A)$	$f(A)$	$f(A)$	$f(A)$	$f(A)$

It is plain that the sum of the coefficients of the quantities thus formed will be the coefficient of x^6 . It appears immediately from formula (β), that each perpendicular column has all its coefficients multiplied by the fluxional coefficient placed underneath it; and by the application of the same formula to the above table, we find with the greatest facility that the coefficient of x^6 in the expansion of $f(A + c'x + c''x^2 + c'''x^3 + \dots)$ is

$$\begin{array}{c} f(A)c^{''''''} + f(A)c'c^{''''} + f(A)\frac{c'^2c^{''''}}{1.2} + f(A)\frac{c'^3c^{''''}}{1.2.3} + f(A)\frac{c'^4c^{''''}}{1.2.3.4} + f(A)\frac{c'^6}{1.2.3.4.5.6} \\ + \frac{c''c^{''''}}{1.2} + \frac{c''^2c^{''''}}{1.2.3} + \frac{c'^2c''c^{''''}}{4} \end{array}$$

And in the same manner may the coefficient of any other power of x be found without any knowledge of those that go before it.

LETTER II.

To the Editor of the Mathematical Repository.

SIR,

I sent you lately a method of expanding any function of a multinomial, which however had the disadvantage of requiring the previous knowledge of another expansion. In *M. Arbogast's* method

method it was deduced from *Taylor's Theorem*, by a rather long process. I have been since a good deal surprised at finding that, notwithstanding the discovery was reserved for a mathematician of our times, it may be accomplished with *as much ease as the common fluxional demonstration of the binomial theorem*.

Let $f(a+x)$ represent any function of $a+x$. If the fluxion be taken with respect to a and x , the fluxional coefficient is plainly the same in both cases; because that coefficient is independent of the magnitude of the fluxion of $a+x$, and remains the same whether this latter be $\dot{a} + \dot{x}$, or $\dot{a}$ or $\dot{x}$. This is analytically expressed thus,

$$\left(\frac{f(a+x)}{\dot{a}} \right) = \left(\frac{f(a+x)}{\dot{x}} \right);$$

Hence it is plain, that

$$\int \left(\frac{f(a+x)}{\dot{a}} \right) \dot{x} = f(a+x).$$

This being premised, let

$$f(A + c'x + c''x^2 + c'''x^3 + \dots + Bx^n + \dots) = B + B_1x + B_2x^2 + \dots + B_nx^n + \dots (x),$$

The first member may be considered as a function of a binomial, the first term being A , and second $c'x + c''x^2 + \dots$. Let B' , B'_1 , B'_2 , &c. represent the fluxional coefficients of B , B_1 , B_2 , &c.

with respect to A supposed variable, and we shall have

$$\frac{f(A + c'x + c''x^2 + c'''x^3 + \dots)}{\dot{A}} = B' + B'_1x + B'_2x^2 + \dots + B'_nx^n + \dots$$

which multiplied by $c'\dot{x} + 2c''x\dot{x} + 3c'''x^2\dot{x} + \dots + nc^{(n)}x^{n-1}\dot{x} + \dots$ will, after the fluent is taken, give $f(A + c'x + c''x^2 + \dots)$; as appears from what was above premised.

We shall have then, by comparison with (x) ,

$$c'B'\dot{x} = B_1x, \quad f(2c''B'_1 + c'B'_2)x\dot{x} = B_2x^2,$$

$$(3c'''B'_1 + 2c''B'_2 + c'B'_3)x^2\dot{x} = B_3x^3, \text{ \&c. and by actual,}$$

ly taking the fluents, the coefficients are expressed by the following recurring terms;

$$B = c'B'$$

$$B = \frac{2c''B'_1 + c'B'_2}{2}$$

$$B = \frac{3c'''B'_1 + 2c''B'_2 + c'B'_3}{3}$$

$$B = \frac{4c'''B' + 3c''B' + 2c'B' + cB'}{4}$$

$$B = \frac{nc^{n-n}B' + (n-1)c^{n-1-(n-1)}B' + \dots + 2c'B' + cB'}{n}$$

But $B = f(A)$ and $B' = \frac{f(A)}{A}$, whence all the rest are known. It is plain that the use of fluents might have been avoided by taking the fluxion of equation (2).

Papcastle,
Decemb. 7, 1808.

I am Sir, Your Obedient Servant,
THOS. KNIGHT

ARTICLE XIX.

Demonstration of a Theorem in the Diophantine Analysis. By Mr. P. BARLOW, of the Royal Military Academy, Woolwich.

To the Editor of the Mathematical Repository.

SIR,

Should the enclosed paper appear to you to merit a place in your next number, its insertion will much oblige

Your's, &c.

August 1, 1808.

P. BARLOW.

THEOREM.

Every integral number whatever, is either a square, or the sum of *two, three, or four* squares.

The above curious property of numbers forms a part of the celebrated theorem of Fermat, viz. "Every number is a triangular number, or the sum of *two, or three* triangular numbers; a square, or the sum of *two, three, or four* squares; a pentagonal, or the sum of *two, three, four, or five* pentagonal numbers; and so on."

This theorem was left by Fermat without a demonstration, nor has it ever been generally demonstrated; although an attempt of that kind has lately been made in one of our monthly publications; the following demonstration, therefore, for the case of squares, may not be unacceptable to some of your English readers. It is not given as original, the only merit that can be claimed is, that of having simplified and extended the demonstration of Le Gendre, by the introduction of the two following propositions.

PROP.

PROP. I.

If A be any prime number, and all the consecutive squares $1^2, 2^2, 3^2, 4^2, \&c. \dots \left(\frac{A-1}{2}\right)^2$, be divided by A , they will each leave a different positive remainder.

For let r^2 and s^2 be any two of those squares, (each of which must necessarily be less than $\frac{1}{2}A$, by the prop,) then if it be possible for them to leave the same remainder after being divided by A , it is evident that A will be a divisor of the difference of those squares, and we shall have $\frac{r^2 - s^2}{A} = \frac{(r+s)(r-s)}{A} = p$ an integral number; but this is impossible, for since $r < \frac{1}{2}A$, and $s < \frac{1}{2}A$, the factors $r+s$, and $r-s$ are each less than A , therefore as A is a prime number, and divides neither of the factors, it cannot divide their product; and consequently, no two of the positive remainders can be equal to each other.

Q. E. D.

Cor. 1. In the same manner it is proved, that the negative remainders are all different from each other.

Cor. 2. Hence also we may see in what cases the positive and negative remainders are equal to each other, for then, it is evident that A will be a divisor of the sum of those squares, and we shall have $\frac{r^2 + s^2}{A} = p$ an integral number. Therefore when A is not a divisor of the sum of two squares, the positive and negative remainders are all different from each other, and include all numbers from 1 to $A-1$.

Cor. 3. When A is not a divisor of the sum of two squares, that is when all the positive and negative remainders are different from each other, then some of each of those remainders, will be *greater*, and some *less* than $\frac{1}{2}A$. For all the consecutive squares under A will be found amongst the positive remainders, some of which must necessarily be *greater*, and some *less* than $\frac{1}{2}A$, and since the positive and negative remainders, include all numbers from 1 to $A-1$, the same, therefore, must evidently be true for the negative remainders.

PROP. II.

If A be any prime number, it is always possible to find four squares w^2, x^2, y^2, z^2 ; such that their sum is divisible by A ; that is the equation $w^2 + x^2 + y^2 + z^2 = 'AA$ is always possible.

When the given prime number A is a divisor of the sum of two squares, the proposition is evident; it will therefore only be necessary

necessary to consider the case in which A is not a divisor of the sum of two squares, and, consequently, when all the remainders of the consecutive squares are different from each other. Cor. 2. Prop. I. Now in this case, we shall find some of the positive remainders *greater*, and some *less* than $\frac{1}{2}A$; and the same of the negative remainders. Cor. 3. Prop. 1. It is, therefore, always possible to find two squares, such that each being divided by A , the positive remainder of one, shall exceed the negative remainder of the other, by unity: and also two others, such that each being divided as before, the negative remainder of one, shall exceed the positive remainder of the other, by unity; that is the equations $w^2 + x^2 - 1 = mA$, and $y^2 + z^2 + 1 = nA$ are always possible.

If this be not evident at first sight, it may be demonstrated thus; let p be the least negative remainder, then $p + 1$ must be found either amongst the positive or negative remainders; now if it be found amongst the positive, we have at once a positive remainder, which exceeds a negative one by unity; and if it be not found amongst the positive, then $p + 1$ is still a negative remainder, and $p + 2$ must be found either amongst the positive or negative remainders, if it be amongst the positive, we shall have a positive remainder, which exceeds a negative one by unity; but if not, then $p + 2$ is still negative and proceeding in the same manner with this last, we must necessarily, (as some of each of the remainders are *greater*, and some *less* than $\frac{1}{2}A$.) arrive at that negative remainder $p + q$, such that $p + q + 1$ shall be a positive remainder; and consequently the equation $w^2 + x^2 - 1 = mA$ is always possible; in the same manner we may demonstrate the possibility of the equation $y^2 + z^2 + 1 = nA$.

Having thus proved the truth of those equations, we have by addition and division $\frac{w^2 + x^2 + y^2 + z^2}{A} = m + n$, or $w^2 + x^2 + y^2 + z^2 = AA \dots\dots\dots Q. E. D.$

PROP. III.

The product of a sum of four squares, by a sum of four squares, is likewise the sum of four squares.

This will be made evident by developing the following formulæ, which will be found identical; viz.

$$\begin{aligned} & (w^2 + x^2 + y^2 + z^2) \times (w'^2 + x'^2 + y'^2 + z'^2) \\ &= (ww' + xx' + yy' + zz')^2 + (wx' - xw' + yz' - zy')^2 \\ &+ (wy' - xz' - yw' + zx')^2 + (wz' + xy' - yx' - zw')^2 \end{aligned}$$

Q. E. D.
PROP.

PROP. IV.

Every prime number is the sum of *two, three, or four* squares.

It has been proved in Prop. 2, that the equation $w^2 + x^2 + y^2 + z^2 = 'AA$ is always possible, where it also appears, that the roots of each of those squares may be less than $\frac{1}{2}A$; the same may be otherwise shewn thus: Since A is a divisor of the formula $(w^2 + x^2 + y^2 + z^2)$, it is also a divisor of the formula $(w - \alpha A)^2 + (x - \beta A)^2 + (y - \gamma A)^2 + (z - \delta A)^2$, where the indeterminates α, β, γ , and δ may always be so assumed, that $(w - \alpha A)^2, (x - \beta A)^2$, &c. shall be less than $(\frac{1}{2}A)^2$; therefore in the following demonstration, whenever it is proved that any number A is a divisor of the sum of four squares, it is taken for granted, that it is also a divisor of four squares, each of which has its root less than $(\frac{1}{2}A)$. This being premised, the demonstration will be as follows:

Since $w^2 + x^2 + y^2 + z^2 = 'AA$, and each of those roots are less than $\frac{1}{2}A$, we have $'AA < \frac{1}{4}A^2$, or $'A < A$. Now if $'A = 1$, it is evident that A will be equal to the sum of the four squares $w^2 + x^2 + y^2 + z^2$, and the proposition is demonstrated.

Let then $'A > 1$, and because $'A$ is a divisor of the formula $(w^2 + x^2 + y^2 + z^2)$, it is also a divisor of $(w - \alpha 'A)^2 + (x - \beta 'A)^2 + (y - \gamma 'A)^2 + (z - \delta 'A)^2$, where each of these roots are less than $\frac{1}{2}'A$.

If now we make $(w - \alpha 'A)^2 + (x - \beta 'A)^2 + (y - \gamma 'A)^2 + (z - \delta 'A)^2 = 'A'A$, we shall have $"A'A < \frac{1}{4}'A^2$, or $"A < 'A$.

Now if by means of the formula at Prop. 3, we multiply the value of $'AA$ by that of $"A'A$, we shall find for a product a sum of four squares, of which each is divisible by $'A^2$, and having performed this division, we shall have $"AA = (A - \alpha w - \beta x - \gamma y - \delta z)^2 + (\alpha x - \beta w + \gamma z - \delta y)^2 + (\alpha y - \gamma w + \delta x - \beta z)^2 + (\alpha z - \delta w + \beta y - \gamma x)^2$; or for the sake of abridging this expression, $"AA = w'^2 + x'^2 + y'^2 + z'^2$. If now $"A = 1$, $A = w'^2 + x'^2 + y'^2 + z'^2$, and the proposition is demonstrated: but if $"A > 1$, we must proceed in the same manner to obtain a new product $"'AA$, expressed by four squares, in which we shall have $"'A < "A$; continuing thus the decreasing series of integral numbers $A, 'A, "A, "'A$, &c. we must necessarily arrive at a term $"^m A$, equal to unity, in which case the prime number A will be expressed by the sum of four squares. Q. E. D.

THEOREM.

Every integral number whatever, is either a square, or the sum of two, three, or four squares.

This is an immediate consequence of the last proposition and of the formulæ at Prop. 3; for every number is either a prime, or produced by the multiplication of prime factors, and since every prime number is of the form $(w^2 + x^2 + y^2 + z^2)$, and the product of two or more such formulæ is still of the same form; it necessarily follows, that every integral number whatever may be represented by $(w^2 + x^2 + y^2 + z^2)$; where it remains to be observed, that no limitation in the course of the demonstration has been made, that can prevent any one, or more of those squares from becoming zero; therefore every integral number whatever, is either a square, or the sum of two, three, or four squares. Q. E. D.

Cor. All that has been proved for integral numbers, is equally true for fractional ones; for every fraction may be reduced to an equivalent fraction having a square denominator; therefore every fraction is of the form

$$\frac{w^2 + x^2 + y^2 + z^2}{m^2} = \frac{w^2}{m^2} + \frac{x^2}{m^2} + \frac{y^2}{m^2} + \frac{z^2}{m^2};$$

this curious property extends, therefore, to every rational number whatever.

ARTICLE XX.

Two letters from Mr. KNIGHT, containing a demonstration of the Binomial Theorem.

LETTER I.

To the Editor of the Mathematical Repository.

SIR,

There is a very simple relation between the coefficients of two successive powers of a binomial: it furnishes the following demonstration of the binomial theorem in the case of whole positive powers; which, however, I do not pretend to be of much merit.

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 \quad \dots \dots \dots (a)$$

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \quad (b)$$

(a) multiplied by 5 gives

$$5(1+x)^4 = 5 + 20x + 30x^2 + 20x^3 + 5x^4, \dots (7)$$

If we put $5 = m$, and represent (β) thus,

$$(1+x)^m = 1 + {}^mA x + {}^m A x^2 + {}^m A x^3 + {}^m A x^4 + {}^m A x^5 \quad (\delta)$$

it is plain by comparing the coefficients of (β) and (γ) , that the latter will become

$$m(1+x)^{m-1} = {}^mA x + 2{}^m A x + 3{}^m A x^2 + 4{}^m A x^3 + \&c. \dots (9)$$

But here let it be particularly observed, that m is not intended to represent any number generally, but only the particular number 5. Multiply (δ) by $1+x$ and there arises

$$m(1+x)^m = \{A + 2A\}x + \{3A + 2A\}x^2 + \{4A + 3A\}x^3 + \dots$$

which equation being compared with (δ) gives

$$A = m, \quad {}^mA = \frac{m(m-1)}{2}, \quad {}^m A = \frac{m(m-1)(m-2)}{2 \cdot 3}, \&c.$$

wherefore in the particular case in which m represents the number 5, we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \&c. \dots (10)$$

It remains to be shewn, that if this equation be true of one value of m , it is so with respect to every other also.

For this purpose multiply (10) by $1+x$ and there will arise

$$(1+x)^{m+1} = 1 + (m+1)x + \frac{(m+1)m}{2}x^2 + \&c.$$

which completes the demonstration.

I am Sir,

Your obedient Servant,

Papcastle,
Oct. 28, 1808.

THOS. KNIGHT.

LETTER II.

To the Editor of the Mathematical Repository.

SIR,

Having before sent you a demonstration of the binomial theorem, in the case of a whole positive exponent, it may not be amiss to shew briefly, that the form is exactly the same in that of a positive or negative fraction. I ought, however, to mention, that the following demonstration, though simpler, does not differ in principle from one given by *Garnier*, in his edition of *Clairaut's Algebra*.

1

Suppose

Suppose that when m is a whole number

$(1+x)^m = A + 'Am + ''Am^2 + \dots$; because $\{(1+x)^m\}^n = (1+x)^{mn}$ we shall have, if n is a whole positive number, and consequently mn ,

$(A + 'Am + ''Am^2 + \dots)^n = A + 'Amn + ''Am^2n^2 + \dots (a)$. Now this equation must plainly be identical, independently of the values of m and n ; let then $m = \frac{s}{n}$, and it becomes

$(A + 'A\frac{s}{n} + ''A\frac{s^2}{n^2} + \dots)^n = A + 'As + ''As^2 + \dots = (1+x)^s$, because s is a whole positive number.

We have then $A + 'A\frac{s}{n} + ''A\frac{s^2}{n^2} + \dots = (1+x)^{\frac{s}{n}}$; whence it appears that by changing, in the expansion of $(1+x)^m$, m into $\frac{s}{n}$, we have that of $(1+x)^{\frac{s}{n}}$.

In equation (a) change n into $-n$, and m into $-\frac{s}{n}$, and it becomes

$(A - 'A\frac{s}{n} + ''A\frac{s^2}{n^2} - \dots)^{-n} = A + 'As + ''As^2 + \dots = (1+x)^s$.

Whence $A - 'A\frac{s}{n} + ''A\frac{s^2}{n^2} - \dots = (1+x)^{-\frac{s}{n}}$: so that, by changing m into $-\frac{s}{n}$, the same expansion is adapted to the case of a negative fraction.

Therefore, in all cases,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2} x^2 + \&c.$$

Papcastle,
Decemb. 7, 1808.

THOMAS KNIGHT.

ARTICLE XXI.

*A Dynamical Principle. By Mr. THOMAS BAZLEY.**To the Editor of the Mathematical Repository.*

SIR,

The following dynamical principle suggested itself to me some time ago, in considering a particular problem. The principle itself is an immediate consequence of the third law of motion; and, notwithstanding its simplicity, it has remained hitherto unnoticed by any writer that I am acquainted with. I have shewn its application to a few known examples, which I think will sufficiently evince the utility and generality of this principle. If it will not occupy too much room in your valuable work, I shall be obliged by its insertion. I am respectfully your's, &c.

Bolton, Oct. 24, 1808.

THOMAS BAZLEY.

If the motion of a body P be any how opposed by the actions of several others connected with it, and the motive forces of these be reduced to the direction of P , and subtracted from its motive force, the remainder is the motive force with which P is urged in consequence of its connexion.

In what follows, I shall confine myself to the determination of the accelerative force; because, when that is known, the circumstances of motion may be determined by the resolution of known equations, (as $\dot{x} = vt$, $t = \dot{x} \div v$, &c.) and consequently the difficulties that occur are purely mathematical.

Ex. 1. Let the bodies p and q be connected by a thread of a given length, and suppose q to be placed on a smooth horizontal plane; and the thread passing perpendicularly over the edge of the plane, let p hang freely: query the force that accelerates p ?

If $g = 16\frac{1}{2}$ feet, then will $2g$ be the measure of the accelerative force of gravity: let x denote the space passed over by p or q , t the time of motion, and v the velocity acquired in that time. Then the motive force of p is $2gp$, and that of q arising from its inertia is $(\dot{v} \div t)q$ as is well known; therefore, by the principle above, the motive force whereby p is now urged, is $2gp - (\dot{v} \div t)q = (\dot{v} \div t)p$ by the known principles, therefore

$\frac{\dot{v}}{t}(p + q) = 2gp$, or $\frac{\dot{v}}{t} = \frac{2gp}{p + q}$ = the accelerative force; and

therefore the circumstances of motion are given by the laws of constant

constant forces. If instead of p we had taken some other moving force, destitute of inertia, which we shall denote by m ; we should have found $m - (\dot{v} \div \dot{t}) q = (\dot{v} \div \dot{t}) \times 0$, because the mass of m is 0, therefore $\frac{\dot{v}}{\dot{t}} \cdot q = m$, or $\frac{\dot{v}}{\dot{t}} = \frac{m}{q}$ = the accelerative force.

Ex. 2. The other notation remaining, let q be placed on an inclined plane, s denoting the sine of inclination; then the motive force of p being $2gp$, the motive force of q arising from inertia is $(\dot{v} \div \dot{t}) q$, and that arising from its gravity is $2g \times s \times q$, and both of these forces resist the motion of p : consequently by the principle

$2g(p - sq) - \frac{\dot{v}}{\dot{t}} q = \frac{\dot{v}}{\dot{t}} p$, and therefore $\frac{\dot{v}}{\dot{t}} = 2g \left(\frac{p - sq}{p + q} \right)$ = the accelerative force.

If $s = 1$, it becomes the case of two bodies connected by a string passing over the edge of the plane when vertical, or if you please, over a pulley; and then $\frac{\dot{v}}{\dot{t}} = \frac{p - q}{p + q} \times 2g$.

Ex. 3. Suppose p to be applied by a thread to a wheel, to elevate q applied by a thread to the axle; and let v denote the velocity of p , b the radius of the wheel, and a that of the axle. Then will $2gp$ be the motive force of p , and $2gq$ = the motive force of q at a arising from gravity; but $b : a :: 2gq : 2gqa \div b$ = motive force of q at b arising from gravity. Again $b : a :: v : av \div b$ = velocity of q , and consequently the motive force at a arising from its inertia is $\dot{v}a \div \dot{t}b$, and $b : a :: \dot{v}a \div \dot{t}b : va^2 \div \dot{t}b^2$ = force of its inertia acting at b : Hence the force with which p descends in consequence of its connexion is

$2g(p - (a \div b) q) - (a^2 \dot{v} \div b^2 \dot{t}) q = (\dot{v} \div \dot{t}) p$; therefore $\frac{\dot{v}}{\dot{t}} \left(\frac{pb^2 + qa^2}{b^2} \right) = 2g(p - \frac{a}{b} q)$, or $\frac{\dot{v}}{\dot{t}} = 2g \cdot \frac{p - \frac{a}{b} q}{p + \frac{a^2}{b^2} q}$ = the

accelerative force.

If we consider p as applied at the distance b to communicate motion, in a horizontal plane, to q placed at the distance a from the same centre: then the gravity of q would not affect the motion

of p and we should have $\frac{\dot{v}}{\dot{t}} = 2gp \div (p + (a^2 \div b^2) q)$; and if we further suppose instead of p , some other power m void of inertia, we shall have $\frac{\dot{v}}{\dot{t}} = \frac{m}{(a^2 \div b^2) q} = \frac{mb^2}{qa^2}$.

Hence

Hence appears the facility with which the properties of circular motion are deducible from our principle.

Ex. 4. The curious problem at page 131 of Simpson's Miscellaneous Tracts (of which an elegant solution is given by Mr. Ivory, at page 22, part II. vol. I. of the Mathematical Repository, New Series) is easily solved by this principle.

PROBLEM. Fig. 26, Pl. D.

Suppose that a thread ACn CA, having two equal weights A, A suspended at the ends thereof, is hung over two tacks C, C, in the same horizontal line; and that to the middle point of the thread (n), equally distant from the tacks, another given weight B is fixed, which is permitted to descend by its own gravity, so as to cause the other two weights, at the same time, to ascend: it is proposed to find the law of the velocity by which the said weights ascend and descend; abstracting from the resistance of the air, the weight of the thread, and the friction of the tacks.

Let v denote the velocity of B (measured by the distance that might be uniformly gone over in one second of time), and let b ($= 32\frac{1}{2}$ feet) = the measure of the velocity, which gravity can generate in a falling body in one second; putting CE = a , En = x , Cn = y ; then we shall have the motive force of B = bxB = force of B to elevate both the weights A, A. The gravity of A is bA , which reduced to the direction nE is $bA(x \div y)$, and consequently, since both weights act against B, the loss of force from their gravity will be $2bA(x \div y)$. Mr. S. shews that the velocity wherewith A ascends is $(v \div y) \sqrt{(y^2 - a^2)}$ whose fluxion is $((y^2 - a^2) y\dot{v} + a^2 v\dot{y}) \div y^2 \dot{x}$, and therefore the resistance of A in direction ACn arising from inertia is $((y^2 - a^2) y\dot{v} + a^2 v\dot{y}) \times A \div y^2 \dot{x}$; and this reduced to the direction nE is $((y^2 - a^2) y\dot{v} + a^2 v\dot{y}) \times A \div y^2 \dot{t}$; and

for both weights it is $\frac{(y^2 - a^2) y\dot{v} + a^2 v\dot{y}}{y^2 \dot{t}} \times 2A$.

Therefore by the principle

$$b(B - \frac{2Ax}{y}) = \frac{(y^2 - a^2) y\dot{v} + a^2 v\dot{y}}{y^2 \dot{t}} \times 2A = \frac{\dot{v}}{\dot{t}} B,$$

or, since $v\dot{t} = \dot{x}$ and $\dot{v} \div \dot{t} = v\dot{v} \div \dot{x}$, by reduction we have

$$\dot{x} - b \cdot \frac{2Ax\dot{x}}{By} = 2A \times \frac{(y^2 - a^2) yv\dot{v} + a^2 v^2 \dot{y}}{By^3} = v\dot{v}, \text{ which}$$

is the very equation deduced by Mr. S. from a process quite different. Hence substituting $y\dot{y}$ for $x\dot{x}$ &c. his conclusions will follow.

Ex.

Ex. 5. The last application we shall make of this principle shall be to the general problem at page 285, of Mr. Atwood's Treatise on Motion.

PROBLEM. Fig. 27, Pl. D.

Let AFC, BGC, be two curves, the planes of which are vertical; and let a line, ACB, be stretched over a fixed pulley, C, by two given weights A and B, of which B preponderates against A, and descends along the curve CGB; it is required to assign the velocity of B, when it has descended through a given perpendicular altitude, and A has ascended through a given altitude in the same time.

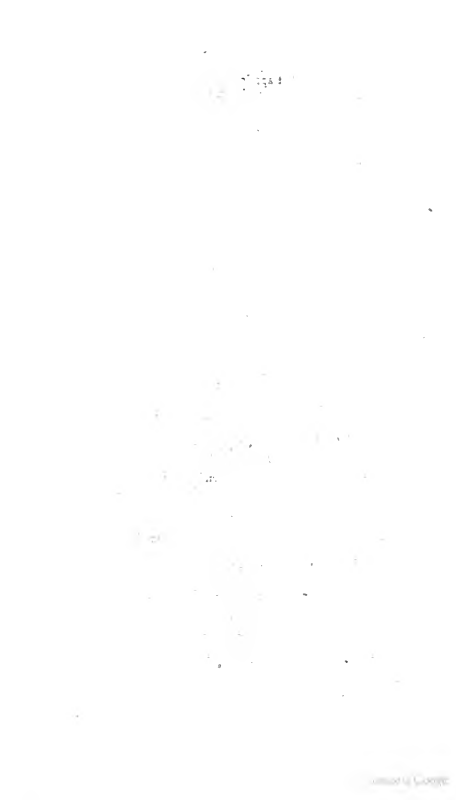
Let BL be the perpendicular altitude through which B has descended, and let AH be the perpendicular altitude through which A has ascended in the same time, and let $BL = y$, $AH = q$. Suppose the weight B to describe in its descent the evanescent arc bB , and during the same time let the other weight A rise through the arc aA . Through b and A draw bo and Al parallel to the horizon; also, through b draw bn perpendicular to CB, and Am perpendicular to Ca, and let $CB = x$, $Bn = \dot{x} = Am$, $Bo = \dot{y}$, $Bb = \dot{s}$, $Aa = \dot{r}$, and $al = \dot{q}$. We find, as Mr. Atwood has done, the motive force of A arising from gravity and reduced to the direction of B $= A\dot{q} \div \dot{s}$; and the resistance from A's inertia reduced to the same direction is $u\dot{u} \div \dot{s}$, theret. by the principle $\frac{B\dot{y}}{\dot{s}} - \frac{A\dot{q}}{\dot{s}} - \frac{Au\dot{u}}{\dot{s}} = \frac{Bv\dot{v}}{\dot{s}}$, where $u = \text{vel. of A}$ and $v = \text{vel. of B}$.

But $\dot{s} : \dot{r} :: v : u = (\dot{r} \div \dot{s})v$, and (since $v\dot{t} = \dot{s}$ and $\dot{v}\dot{t} = \ddot{s}$, &c. therefore $\dot{s} : \dot{r} :: v : u = (\dot{r} \div \dot{s})v$; whence $u\dot{u} = (\dot{r} \div \dot{s})\dot{v}\dot{v}$, therefore $B\dot{y} - A\dot{q} = Au\dot{u} + Bv\dot{v} = (A\dot{r} \div \dot{s} + B\dot{s})\dot{v}\dot{v} \div \dot{s}$; consequently the accelerating force of B $= \frac{v\dot{v}}{\dot{s}} = \frac{(B\dot{y} - A\dot{q})\dot{s}}{A\dot{r} + B\dot{s}}$ which is the identical expression found by Mr. Atwood.

End of the Second Part of the Second Volume.

ERRATUM.

Part 2nd. page 53, line 11 from the bottom, for "infinite" read "finite".





NEW SERIES
OF THE
MATHEMATICAL REPOSITORY.

VOL. II. PART III.

MATHEMATICAL MEMOIRS, EXTRACTED
FROM WORKS OF EMINENCE.

ARTICLE I.

A MEMOIR ON ELLIPTIC TRANSCENDENTALS.

Containing easy methods of comparing and valuing these Transcendentals, which include elliptic arches, and which frequently occur in the application of the Integral Calculus.*

(Read to the Ci-devant Academy of Sciences, in April, 1792.)

By ADRIEN MARIE DE GENDRE, Member of the National
Institute of France, and F. R. S. L.

Many integrals may be determined solely by the aid of arches of a circle and logarithms, which are the most simple transcendentials; but to extend the application of the integral calculus, recourse must be had to transcendentials of a more complex form. It is by examining with care the nature and properties of these, at least such as are of more frequent use, that we can simplify considerably the results of the theory, and render it more convenient in practice.

Next

* LACROIX observes, in a note to his valuable Treatise on the Integral Calculus that this interesting *Memoir*, which makes no part of any academical collection, was become very rare, there having been but few copies of it originally printed. It is inserted here at the request of several eminent Mathematicians.

Next to arches of a circle and logarithms, elliptic arches are one of the most simple transcendentals, and they might be made in some measure a new instrument of calculus, if their properties were once commonly known, and we possessed easy methods of estimating them accurately. Upon this subject, we have proposed some new views, in the Memoirs of the Academy of Sciences of Paris, for the year 1786.

But if elliptic arches add greatly to the means of analysis, more especially when it is considered that hyperbolic arches may be easily deduced from them, they are, however, inadequate to some inquiries which are in some respects but little complicated, such as the surface of an oblique cone, the motion of rotation of a body which is not impelled by any accelerating force, &c. These and innumerable others depend upon the

integral $\int \frac{Pdx}{R}$, in which P is a rational function of x , and R a

radical of the form $\sqrt{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4)}$, α, β, γ , &c. being constant. Now a careful examination of the nature of this integral shews, that it presents three distinct species of transcendentals. The first and second may be expressed by elliptic arches, the third is more complex, but it has such an analogy with the two others, that all three may be considered as constituting but one and the same order of transcendentals, the next to arches of a circle and logarithms.

As the two first species are expressible by arches of ellipses, they might have been included in one, however we have thought proper to distinguish them, observing that what we call the first species may be determined by the second, but not the second by the first: hence it appears that the first species considered analytically is more simple in its nature than the other; and therefore, that elliptic arches are not the most simple transcendental that occurs after arches of a circle and logarithms. Upon the whole, the three species of transcendentals, which we are considering, are so related to each other, that we have judged it proper to give them the common name of *Elliptic Transcendentals*.

We propose in this memoir to explain the nature and properties of this order of transcendentals, so as to render its use easy in the application of the integral calculus. Some of the methods and results which we shall explain are already known to geometers; we have brought into one point of view all that has hitherto been published relating to this theory, but we have, at the same time, endeavoured to give it a greater degree of perfection.

A general

A general idea of the different kinds of transcendentials, contained in the integral formula $\int \frac{Pdx}{R}$.

1. We shall put P for any rational function whatever of x ; and R for the radical $\sqrt{(x + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4)}$; this radical remaining the same, an infinity of different values may be given to P, but there will not thence result an infinity of different transcendentials; for we shall shew that the integral $\int \frac{Pdx}{R}$ may always, by partial integrations, be reduced to an algebraic part, plus a certain number of transcendentials, which are all of the same form and nature.

First, let us suppose that P is an integer function of x , such, that $P = A + Bx + Cx^2 + \dots + Kx^k$. If, in order to abridge, we represent the integral $\int \frac{x^m dx}{R}$ by Π^m , it is evident that $\int \frac{Pdx}{R} = A\Pi^0 + B\Pi^1 + C\Pi^2 + \dots + K\Pi^k$;

now, by taking the differential of the quantity $x^{m-3} R$, and then returning from the differential to the integral, we get this formula

$$x^{m-3} R = (m-3)\alpha \Pi^{m-4} + (m-\frac{1}{2})\beta \Pi^{m-3} + (m-2)\gamma \Pi^{m-2} \\ + (m-\frac{1}{2})\delta \Pi^{m-1} + (m-1)\epsilon \Pi^m;$$

hence it follows, that Π^m , Π^{m-1} , &c, may be reduced successively to lower degrees; and that the reduction may be repeated till the quantities be all of a lower degree than Π^3 , for the reduction of Π^3 is also possible, because, that in the case of $m=3$, the term which would seem to contain Π^{-1} vanishes. Therefore, P

being any integer function of x , the integral $\int \frac{Pdx}{R}$ may always be reduced to an algebraic quantity, together with a transcendental, which in every case will be of this form

$$\int (A + Bx + Cx^2) \frac{dx}{R},$$

2. Let us now consider the formula in its most general form, and let P be any rational function whatever of x ; we proceed

ceed as in the integration of the rational fraction Pdx , and in the first place we separate the integer part of it, which may be treated as has been already explained. We next decompose the remaining part into several partial fractions, according to the number of factors in the denominator; these constitute so many terms of the whole integral, and each term may be represented

by the general expression $N \int \frac{dx}{(1+nx)^k R}$. But we shall

immediately shew that it is easy to reduce all the terms of this form to similar terms in which $k = 1$. And in the first place, we observe that if the denominator, instead of being complex,

were simply x^k , the integral $\int \frac{dx}{x^k R}$, and all similar to it in

which $k > 1$, might be determined by means of the integral

$\int \left(\frac{A}{x} + B + Cx + Dx^2 \right) \frac{dx}{R}$; for this purpose, it would only

be necessary to give negative values to $m-4$, in the formula of article 1.

Let us now return to the general term $\int \frac{dx}{(1+nx)^k R}$, which

we shall call r^k . If we put $1+nx = \omega$, and take the co-efficients, $\alpha', \beta', \&c.$ so as to have the identical equation

$$\alpha + \beta \left(\frac{\omega-1}{n} \right) + \gamma \left(\frac{\omega-1}{n} \right)^2 + \delta \left(\frac{\omega-1}{n} \right)^3 + \epsilon \left(\frac{\omega-1}{n} \right)^4 \\ = \alpha' + \beta' \omega + \gamma' \omega^2 + \delta' \omega^3 + \epsilon' \omega^4,$$

we shall find by the differentiation of the quantity $\omega^{-k+1} R$ this general formula

$$\frac{-R}{n(1+nx)^{k-1}} = (k-1)\alpha' r^k + (k-\frac{3}{2})\beta' r^{k-1} + (k-2)\gamma' r^{k-2} \\ + (k-\frac{1}{2})\delta' r^{k-3} + (k-3)\epsilon' r^{k-4}.$$

Hence it follows, that the quantities $r^2, r^3, \&c.$ may be determined by means of r^1, r^0, r^{-1}, r^{-2} ; but $r^0 = \int \frac{dx}{R},$

$r^{-1} = \int (1+nx) \frac{dx}{R}, r^{-2} = \int (1+nx)^2 \frac{dx}{R}$. Therefore,

in general the formula $\int \frac{dx}{(1+nx)^k R}$, and all others similar to it,

in

in which k is greater than unity, are reducible to an algebraic part, together with an integral of this form

$$\int \left(\frac{A}{1+nx} + B + Cx + Dx^2 \right) \frac{dx}{R}.$$

3. Hence we conclude, that P being put for any rational function whatever of x , the integral $\int \frac{Pdx}{R}$ is always resolvable into three principal parts, the first is algebraic, the second has this form $\int (A + Bx + Cx^2) \frac{dx}{R}$, and the third contains one or more terms of the form $N \int \frac{dx}{(1+nx)R}$, where the coefficient n may be either real or imaginary.

It appears by this analysis that the number of transcendentals contained in the formula $\int \frac{Pdx}{R}$ is very limited, for there are only two principal species, the one having the form $\int (A + Bx + Cx^2) \frac{dx}{R}$, and the other the form $\int \frac{dx}{(1+nx)R}$. I even observe, that as often as n is real, the second species is contained in the first, and may be immediately reduced to it by making $1 + nx = \frac{1}{z}$.

These general reductions being once understood we proceed in a different manner to attain a more precise knowledge of the same transcendentals.

Of the way in which the odd powers of the variable quantity under the radical may be made to disappear.

4. The radical being always $\sqrt{(x + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4)}$, let us suppose that the quantity under the sign is resolved into two real factors $\zeta + 2\alpha x + \theta x^2$, $\lambda + 2\mu x + \nu x^2$. These factors ought to have the same sign that the radical may be real; therefore we may suppose that

$$\zeta + 2\alpha x + \theta x^2 = (\lambda + 2\mu x + \nu x^2) y^2.$$

Hence we find

$$x = \frac{\mu y^2 - \alpha + \sqrt{\{(\mu y^2 - \alpha)^2 + (\lambda y^2 - \zeta)(\theta - \nu y^2)\}}}{\theta - \nu y^2};$$

and,

and, in order to abridge, Y is put for the radical of this expression, we have $\frac{dx}{R} = \frac{dy}{Y}$. The value of x being substituted

in P , it appears that the differential $\frac{Pdx}{R}$, or $\frac{Pdy}{Y}$ may be always resolved into two parts, the one rational and integrable by the common rules, and the other affected by the radical Y , but such, that only the even powers of y are found either under the radical or without it. Thus the integration of the formula $\frac{Pdx}{R}$ is always reducible to that of a formula of the same kind, but in which P and R contain only the even power of x .

5. This method is general; however, as the value of x which ought to be substituted in P is a little complex, it may not be useless to shew that the same end may be obtained by a much more simple substitution, which consists in putting $x = \frac{p+qy}{1+y}$, p and q being two indeterminate constant quantities.

Resuming now the factors $\lambda + 2\mu x + \nu x^2$, $\lambda + 2\mu x + \nu x^2$, and substituting in each of them the value of x , we may leave out of consideration the common denominator, which goes out of the radical, then, that the odd powers of y may disappear in the numerators, it is necessary that

$$\begin{aligned} + n(p+q) + 6pq &= 0. \\ \lambda + \mu(p+q) + \nu pq &= 0. \end{aligned}$$

These two equations give rational values of $p+q$ and pq , but that p and q may be real, it is necessary that $(p+q)^2 - 4pq$, or $(p-q)^2$ be positive. Now we may distinguish two cases.

1st. If the factors of the quantity $\alpha + \beta x + \gamma x^2 + \&c.$ are not all real, let it be supposed that $\lambda + 2\mu x + \nu x^2$ contains two of them which are imaginary, and consequently, that $\lambda > \mu^2$. But the second equation containing p and q , gives

$$(p-q)^2 = \left(\frac{\lambda + \nu pq}{\mu} - \frac{2\mu}{\nu} \right)^2 + \frac{4\lambda\nu - 4\mu^2}{\nu^2};$$

therefore, in the first case, the second member of the equation is positive, and the values of p and q are real.

2d. If all the factors of the quantity $\alpha + \beta x + \gamma x^2 + \&c.$ be real, and if that in resolving it into two factors of the second degree, as has been done, the values of p and q are not real, then it is to be observed that there are two other ways

ways in which a quantity of the fourth degree may be resolved into two factors of the second, and we may be assured that these two new combinations will give real values of p and q . This we shall now demonstrate.

Let a, b, c, d , be the four values of x which are found by making $R = 0$, the equations which determine p and q may be written thus,

$$\begin{aligned} ab - \frac{1}{2}(a+b)(p+q) + pq &= 0 \\ cd - \frac{1}{2}(c+d)(p+q) + pq &= 0. \end{aligned}$$

Hence we get

$$\begin{aligned} p+q &= \frac{ab-cd}{a+b-c-d} \\ \left(\frac{p-q}{2}\right)^2 &= \frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}. \end{aligned}$$

But it is always in our power to render the denominator of this last quantity positive; for suppose that a, b, c, d , are written in the order of their magnitudes, the negative quantities coming after the positive, then the differences $a-b, a-c$, &c. shall all be positive, and $p-q$ real. We shall also have $p-q$ real, if for a and b we take the greatest and least of the four roots a, b, c, d . Finally, the third combination gives $p-q$ imaginary.

Therefore, by the substitution, $x = \frac{p+qy}{1+y}$, it is always

possible to cause the odd powers of the variable quantity under the radical to disappear, and at the same time the two factors under the new radical are real and of the form $f+gy^2$, which is a new and important advantage, and which cannot be obtained by the first substitution.

Observe that there are two particular cases to be examined.

1st. When one of the roots a and b is equal to one of the roots c and d .

2d. When $a+b=c+d$. In the first case the radical R becomes more simple, and the integral depends only upon arches of a circle and logarithms. With respect to the second case, a simple permutation is sufficient to prevent $a+b$ from being equal to $c+d$, but this circumstance may be employed to facilitate the transformation. For the two factors under the radical having then the form $\lambda + \nu(x^2 + 2mx)$, $\zeta + \theta(x^2 + 2mx)$, it is evident that if we put $x+m=y$, the uneven powers of the variable quantity disappear.

Reduction

Reduction of the differential $\frac{Pdx}{R}$ to the form $\frac{Qd\phi}{\sqrt{(1-c^2 \sin.^2 \phi)}}$.

6. Since by a previous preparation we can eliminate the odd powers of the variable quantity under the radical, we shall in future put $R = \sqrt{(x + \beta x^2 + \gamma x^4)}$, we shall also suppose that P is an even function of x , for whatever be the rational function P , we can always make $P = M + Nx$, M and N being even functions of x : now by making $x^2 = y$, the part $\frac{Nx dx}{R}$ is reducible to the common rules, and thus all the difficulty is thrown upon the integration of the formula $\frac{M dx}{R}$, in which M is an even function of x .

This being understood, we proceed to prove, by enumerating the different cases, that the differential $\frac{dx}{R}$ may always be reduced to the form $\frac{md\phi}{\sqrt{(1-c^2 \sin.^2 \phi)}}$, c being less than unity.

FIRST CASE. Suppose that the factors of $\alpha + \beta x^2 + \gamma x^4$ are imaginary; let this quantity be expressed by $\lambda^2 + 2\lambda\mu x^2 \cos. \theta + \mu^2 x^4$, λ and μ being positive, and $\cos. \theta$ either positive or negative. We make $x = \sqrt{\left(\frac{\lambda}{\mu}\right)} \tan. \frac{1}{2} \phi$, then, taking $c = \sin. \frac{1}{2} \theta$, we have the transformed equation

$$\frac{dx}{R} = \frac{1}{2\sqrt{\lambda\mu}} \cdot \frac{d\phi}{\sqrt{(1-c^2 \sin.^2 \phi)}}.$$

SECOND CASE. Let $\alpha + \beta x^2 + \gamma x^4 = m^2 (1 + p^2 x^2) (1 - q^2 x^2)$, the limit of x being $\frac{1}{q}$; by putting $x = \frac{1}{q} \cos. \phi$ and $\frac{p^2}{p^2 + q^2} = c^2$ we get

$$\frac{dx}{R} = \frac{c}{mp} \cdot \frac{-d\phi}{\sqrt{(1-c^2 \sin.^2 \phi)}}.$$

If we wish the transformed equation to be positive, we must make $\cot. \phi = \sqrt{(1-c^2)} \tan. \psi$, and hence we have immediately $x^2 = \frac{\sin.^2 \psi}{q^2 + p^2 \cos.^2 \psi}$, and the transformed equation

$$\frac{dx}{R} = \frac{c}{mp} \cdot \frac{d\psi}{\sqrt{(1-c^2 \sin.^2 \psi)}}.$$

THIRD

THIRD CASE. Let $x + \beta x^2 + \gamma x^3 = m^2(1 + p^2 x^2)(x^2 - q^2)$;
we put $x = \frac{q}{\cot. \varphi}$, $\frac{1}{1 + p^2 q^2} = c^2$, and hence we get

$$\frac{dx}{R} = \frac{c}{m} \cdot \frac{d\varphi}{\sqrt{(1 - c^2 \sin.^2 \varphi)}};$$

FOURTH CASE. Let $x + \beta x^2 + \gamma x^3 = m^2(1 + p^2 x^2) \times$
 $(1 + q^2 x^2)$, and let $p > q$, by putting $x = \frac{\tan. \phi}{p}$, and $\frac{p^2 - q^2}{p^2} = c^2$,
we have

$$\frac{dx}{R} = \frac{1}{mp} \cdot \frac{d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}}.$$

FIFTH CASE. Let $x + \beta x^2 + \gamma x^3 = m^2(1 - p^2 x^2) \times$
 $(1 - q^2 x^2)$, and let $p > q$, then, putting $px = \sin. \phi$, and $\frac{q}{p} = c$,
the transformed equation becomes

$$\frac{dx}{R} = \frac{1}{mp} \cdot \frac{d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}}.$$

This formula is applicable from $x = 0$ to $x = \frac{1}{p}$; the radical R is imaginary from $x = \frac{1}{p}$ to $x = \frac{1}{q}$; but it becomes again real from $x = \frac{1}{q}$ to $x = \infty$. In the last case it is necessary to write $R^2 = m^2(p^2 x^2 - 1)(q^2 x^2 - 1)$; then, making $qx = \frac{1}{\sin. \phi}$, and $\frac{q}{p} = c$, the transformed equation becomes $\frac{-d\phi}{mp \sqrt{(1 - c^2 \sin.^2 \phi)}}$. If we wish to have it positive, we must make, as before, $\cot. \phi = \sqrt{(1 - c^2)} \tan. \psi$, which immediately gives $x^2 = \frac{1 - c^2 \sin.^2 \psi}{q^2 \cos.^2 \psi}$, and the transformed equation is

$$\frac{dx}{R} = \frac{1}{mp} \cdot \frac{d\psi}{\sqrt{(1 - c^2 \sin.^2 \psi)}},$$

a formula perfectly similar to the first, from which it follows that the integral of $\frac{dx}{R}$, in the case of $x > \frac{1}{q}$ may always be found from the same integral taken upon the supposition of $x < \frac{1}{p}$.

SIXTH CASE. Lastly. Let $\alpha + \beta x^2 + \gamma x^4 = m, (x^2 - q^2) \times (p^2 - x^2)$; then x must be contained between p and q . Let

$p > q$, and assume $x = \frac{q}{\cos. \psi}$; then we have immediately

$$\frac{dx}{R} = \frac{d\psi}{m\sqrt{p^2 - q^2 - p^2 \sin.^2 \psi}}.$$

In this formula the angle ψ has a limit; but that we may introduce into it one which is indefinite, let $\sin. \psi = c \sin. \phi$,

and $c^2 = \frac{p^2 - q^2}{p^2}$, thus the formula is transformed to

$$\frac{dx}{R} = \frac{1}{mp} \cdot \frac{d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}},$$

to which we might have come directly by making $x^2 =$

$$\frac{q^2}{1 - c^2 \sin.^2 \phi}.$$

7. It may now be remarked that, setting aside the first case, the values which must be given to x^2 , so as to produce the re-

quired reduction, have always the form $\frac{A + B \sin.^2 \phi}{C + D \sin.^2 \phi}$, where

the coefficients are constant. Therefore, it is only necessary

to substitute this value of x^2 in P, and the formula $\int \frac{P dx}{R}$ shall

be transformed into another $\int \frac{Q d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}}$, in which c is

less than unity, and Q is a rational and even function of $\sin. \phi$, which contains $\sin. \phi$ to the same degree that P contains x .

We set aside the first case, because of the form of the value of x being a little different, and also, because that by following the method of article 5, this case is avoided, seeing that we always get real factors. But in another article we shall return to the case of imaginary factors.

Development of the formula $\int \frac{Q d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}}$.

8. We shall henceforth denote the radical $\sqrt{(1 - c^2 \sin.^2 \phi)}$ by Δ , and also by $\Delta(\phi)$ and by $\Delta : (c, \phi)$, according as it is considered as a function of ϕ only, or as a function of c and ϕ .

Let us in the first place consider the case in which Q is an integer function, and let $Q = A + B \sin.^2 \phi + C \sin.^4 \phi + \&c.$
If

If, for the sake of abridging, we represent the integral $\int \frac{\sin.^{2k} \phi d\phi}{\Delta}$ by Z^k , we have $\int \frac{Q dx}{\Delta} = AZ^0 + BZ^1 + CZ^2 + \&c.$

But, by taking the differential of the quantity $\Delta \cos. \phi \sin.^{2k-3} \phi$, we readily obtain the formula $\Delta \cos. \phi \sin.^{2k-3} \phi = (2k-3) \times Z^{2k-4} - (1+c^2)(2k-2)Z^{2k-2} + c^2(2k-1)Z^{2k}$; Hence it follows that $Z^4, Z^6, \&c.$ may be expressed by means of Z^0 and Z^2 , and therefore, that in the case when Q is an integer function, the integral $\int \frac{Q d\phi}{\Delta}$ is equal to an algebraic quantity, plus an integral of the form $\int (A + B \sin.^2 \phi) \frac{d\phi}{\Delta}$.

9. Now let Q be any rational function of $\sin.^2 \phi$. After having separated the integer part, and reduced it, in the manner which has been explained, the developement of the fractional part may always be so ordered, that every partial fraction shall be of the form $\frac{N}{(1+n \sin.^2 \phi)^k}$, n and N being constant coefficients, real or imaginary. We must now endeavour to reduce the formula $\int \frac{d\phi}{(1+n \sin.^2 \phi)^k \Delta}$, and such like, to the most simple formulas of the same kind. But if we make for a little, $\sin \phi = x$, the formula becomes

$$\int \frac{dx}{(1+nx^2)^k \sqrt{(1-(1+c^2)x^2+c^2x^4)}}.$$

Let us consider, for the sake of greater generality, the formula

$$\Pi^k = \int \frac{dx}{(1+nx^2)^k R},$$

in which R represents the radical $\sqrt{(a+\beta x^2+\gamma x^4)}$. By differentiation, we find

$$\frac{xR}{(1+nx^2)^{k-1}} = (2k-2)\left(a-\frac{\beta}{n}+\frac{\gamma}{n^2}\right)\Pi^{k-(2k-3)}\left(a-\frac{2\beta}{n}+\frac{3\gamma}{n^2}\right) \times \\ \Pi^{k-1} + (2k-4)\left(-\frac{2\beta}{n}+\frac{3\gamma}{n^2}\right)\Pi^{k-2} - (2k-5)\frac{\gamma}{n^2}\Pi^{k-3}.$$

β 2

Hence

Hence it follows that the term Π^k , and all such, in which k is greater than unity, may be expressed in part algebraically, and in part by the quantities Π^1 , Π^0 , Π^{-1} , which are the most simple of their kind. It may even be observed that if $1 + nx^2$ be a divisor of the quantity under the radical, in which case we have $\alpha - \frac{\beta}{n} + \frac{\gamma}{n^2} = 0$, then the preceding formula has a term left, and thus the quantity Π^k depends only on these two Π^0 and Π^{-1} .

Therefore the determination of Π^k depends in general on the formula

$$\int \left(\frac{A}{1 + nx^2} + B + Cx^2 \right) \frac{dx}{R},$$

and in particular, when $1 + nx^2$ is a divisor of R^2 , the denominator may be made to disappear, and the determination of Π^k will then depend only on the formula $\int (B + Cx^2) \frac{dx}{R}$.

10. The application of this result to the integral

$\int \frac{d\varphi}{(1 + n \sin.^2 \varphi)^k \Delta}$ shews, that in general this integral depends upon the following

$$\int \left(\frac{A}{1 + n \sin.^2 \varphi} + B + C \sin.^2 \varphi \right) \frac{d\varphi}{\Delta}.$$

And in the particular case, in which $1 + n \sin.^2 \varphi$ is a factor of $\cos.^2 \varphi \Delta^2$, the integral may be freed from the denominator, and will then depend entirely on the integer formula

$$\int (B + C \sin.^2 \varphi) \frac{d\varphi}{\Delta}.$$

This particular case occurs when $n = -1$, also when $n = -c^2$; then the formulas are $\int \frac{d\varphi}{\cos.^{2k} \varphi \Delta}$ and $\int \frac{d\varphi}{\Delta^{2k+1}}$, and indeed it is easy to reduce both to an algebraic part together with an integral of the form $\int (A + B \sin.^2 \varphi) \frac{d\varphi}{\Delta}$. We may add to these two cases that of $\int \frac{d\varphi}{\sin.^{2k} \varphi \Delta}$ which is susceptible of a similar reduction, and which may be effected by the formula of article 8.

It

It follows from this analysis that the general formula $\int \frac{Q d\varphi}{\Delta}$ when reduced to a convenient form contains; 1°, an algebraic part; 2°, an integral of the form $\int (A + B \sin.^2 \varphi) \frac{d\varphi}{\Delta}$; 3°, one or more parts of the form $\int \frac{N}{1 + n \sin.^2 \varphi} \cdot \frac{d\varphi}{\Delta}$, in each of which the coefficients N and n have any values whatever, real or imaginary.

Definition of Elliptic Transcendentals and their division into three Species.

11. Since the transcendentals we have considered are always reducible to these two forms $\int (A + B \sin.^2 \varphi) \frac{d\varphi}{\Delta}$, $\int \frac{d\varphi}{(1 + n \sin.^2 \varphi) \Delta}$, it is evident that they are comprehended in the general formula

$$H = \int \frac{A + B \sin.^2 \varphi}{1 + n \sin.^2 \varphi} \cdot \frac{d\varphi}{\Delta}.$$

We shall in future call the integrals contained in this formula *elliptic functions* or *elliptic transcendentals*. We shall suppose the transcendental H to vanish, or to begin, when $\varphi = 0$; and its extent to be determined by the variable arc φ , which we shall call its *amplitude*; the constant quantity c , which is always less than unity, shall be called the *modulus*; and, lastly, $\sqrt{(1 - c^2)}$ which we shall always denote by b , shall be called the *complement of the modulus*.

The quantity H , when φ is considered as the only variable quantity in it, may be denoted by $H(\varphi)$; if however the modulus and amplitude be both considered as variable, it may be denoted by $H(c, \varphi)$; and so on, if its coefficients be supposed to change their values.

12. If every value of H , from $\varphi = 0$ to $\varphi = 90^\circ = \frac{\pi}{2}$, were known, we could easily determine the value of the transcendental for any other amplitude: for let $\varphi = i\pi \pm \alpha$, i being a whole number, and α an arch less than $\frac{\pi}{2}$, then,

$$H(\varphi) = 2i H\left(\frac{\pi}{2}\right) \pm H(\alpha).$$

In

In this respect the function H , agrees with elliptic arcs, and in general with the arcs of all oval curves composed of four equal and similar parts, for any arc of these, however great, and containing even several circumferences, may always be expressed without difficulty, by the fourth of the curve and a certain portion of that fourth. The determinate function $H\left(\frac{\pi}{2}\right)$ may therefore be considered as the unit of the functions H , and we shall denote it by H_1 .

It does not appear that the function H , taken in all its generality, can be reduced to elliptic arcs; this can only happen when $n = 0$, or when by some substitution the denominator $1 + n \sin.^2 \varphi$ is made to disappear, a reduction which we do not suppose to be always possible.

Thus the term *elliptic function* is in some respects improper; we retain it however because of the great analogy which is found to subsist between the properties of this function, and those of elliptic arcs.

13. When $n = 0$, or when by a suitable substitution the denominator $1 + n \sin.^2 \varphi$ vanishes, then the function H is rendered considerably more simple. In this case it will be proper to denote it by another character G , thus we shall henceforth make

$$G = \int \frac{(A + B \sin.^2 \varphi) d\varphi}{\sqrt{(1 - c^2 \sin.^2 \varphi)}}.$$

The function G , as we shall hereafter demonstrate, may always be readily expressed by two elliptic arcs, and with respect to elliptic arcs themselves, they are represented by the formula

$$E = \int d\varphi \sqrt{(1 - c^2 \sin.^2 \varphi)},$$

concerning which, see *Memoirs of the Academy for 1786*.

In short, if regard be had to analytical simplicity, the function $\int \frac{d\varphi}{\sqrt{(1 - c^2 \sin.^2 \varphi)}}$ is very remarkable, and it is proper to design it by a particular symbol, we shall therefore make

$$F = \int \frac{d\varphi}{\sqrt{(1 - c^2 \sin.^2 \varphi)}};$$

this function F may much rather be expressed by two elliptic arcs, but it may also be expressed by the arc E , and its partial difference relative to the modulus c ; for it is evident that

$$F = E - c \frac{dE}{dc}.$$

We

We even believe that this expression would be the most convenient for the numerical calculation of integrals, if there were tables of elliptic arcs, which should also exhibit the values of the coefficient $\frac{dE}{dc}$ at the side of each arc.

However this may be, the quantity F may certainly be expressed by E , but not E by F , and much less G by F , and this appears to me to be a sure proof that the function F , considered analytically, is more simple than elliptic arcs. If we judge of hyperbolic arcs in the same way it will appear that they are absolutely of the same nature as elliptic arcs, for like as an hyperbolic arc can be expressed by two elliptic arcs, so, on the contrary, an elliptic arc can be expressed by two hyperbolic arcs, this will be evident from the values we shall afterwards give for these arcs, and which may also be found in the Memoirs for 1786 already quoted.

14. There are therefore three distinct species of elliptic functions or transcendentials. The first, and most simple, is represented by the formula

$$F = \int \frac{d\varphi}{\Delta}.$$

The second is represented by the formula

$$G = \int (A + B \sin.^2 \varphi) \frac{d\varphi}{\Delta}.$$

This contains, as a particular case, the elliptic arc $E = \int d\varphi \Delta$, and the hyperbolic arc $T = \Delta \tan. \varphi - \int (1 - \sin.^2 \varphi) \frac{c^2 d\varphi}{\Delta}$. Lastly, the third, and most complex, is expressed by the formula

$$H = \int \frac{A + B \sin.^2 \varphi}{1 + n \sin.^2 \varphi} \cdot \frac{d\varphi}{\Delta}.$$

In this last the coefficients A, B, n may be imaginary, but then the formula H will be accompanied by a similar formula, which will differ from it only by the sign of $\sqrt{-1}$, and the sum of the two will always give a real result.

We might also, for the sake of avoiding imaginary quantities, consider the formula

$$K = \int \frac{A + B \sin.^2 \varphi + C \sin.^4 \varphi}{1 + 2n \cos. \alpha \sin.^2 \varphi + n^2 \sin.^4 \varphi} \cdot \frac{d\varphi}{\Delta};$$

as a distinct species of elliptic functions, but we do not think it necessary to admit this fourth species of transcendentials; and we shall confine our attention to the three others, because the calculations by which the value of H is determined are the same whether

whether n be real, or imaginary. It is however indispensably necessary that the modulus c , and the amplitude ϕ be both real, and that at the same time c be less than unity.

Comparison of elliptic functions of the first kind.

15. Every geometer knows the complete algebraic integral which Euler has given of the equation

$$\frac{dx}{\sqrt{(x+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4)}} + \frac{dy}{\sqrt{(x+\phi y+\gamma y^2+\delta y^3+\varepsilon y^4)}} = 0.$$

This integral will be of great use in our investigations. And in the first place, in consequence of the foregoing transformations, this equation may, without losing its generality, be put under the form

$$\frac{d\phi}{\sqrt{(1-c^2 \sin.^2 \phi)}} + \frac{d\psi}{\sqrt{(1-c^2 \sin.^2 \psi)}} = 0;$$

its integral then is $F(\phi) + F(\psi) = \text{const.}$ But the same integral found by Euler's method, and reduced to the most simple form, is $\text{cos. } \phi \text{ cos. } \psi - \sin. \phi \sin. \psi \sqrt{(1-c^2 \sin.^2 \mu)} = \text{cos. } \mu \dots (a')$. Here μ is the arbitrary constant quantity, and as it appears that putting $\phi = 0$, then $\psi = \mu$, herefore the first integral becomes

$$F(\phi) + F(\psi) = F(\mu).$$

Therefore, as often as the angles ϕ, ψ, μ have among themselves the relation expressed by the equation (a') , the function $F(\mu)$ shall be equal to the sum of the two $F(\phi), F(\psi)$. We shall here leave the consideration of the differential equation, that we may examine the consequences which follow from this result, with respect to elliptic functions of the first species.

16. Having given any two elliptic functions $F(\phi), F(\psi)$, a third function $F(\mu)$ may be found equal to their sum; for this purpose it will be necessary to deduce the value of μ from the equation (a') . But, by denoting always $\sqrt{(1-c^2 \sin.^2 \phi)}$ by Δ or $\Delta(\phi)$, and the like radical of ψ by $\Delta(\psi)$, we find

$$\text{cos. } \mu = \frac{\text{cos. } \phi \text{ cos. } \psi - \Delta(\phi) \Delta(\psi) \sin.^2 \phi \sin.^2 \psi}{1 - c^2 \sin.^2 \phi \sin.^2 \psi},$$

$$\sin. \mu = \frac{\Delta(\psi) \sin. \phi \text{ cos. } \psi + \Delta(\phi) \sin. \psi \text{ cos. } \phi}{1 - c^2 \sin.^2 \phi \sin.^2 \psi},$$

$$\Delta(\mu) = \frac{\Delta(\phi) \Delta(\psi) - c^2 \sin. \phi \sin. \psi \text{ cos. } \phi \text{ cos. } \psi}{1 - c^2 \sin.^2 \phi \sin.^2 \psi},$$

$$\tan. \mu = \frac{\Delta(\psi) \tan. \phi + \Delta(\phi) \tan. \psi}{1 - \Delta(\phi) \Delta(\psi) \tan. \phi \tan. \psi}.$$

It may be remarked with regard to this last formula, that if two angles φ and ψ be taken, such, that

$$\tan. \varphi' = \Delta(\psi) \tan. \varphi, \tan. \psi' = \Delta(\varphi) \tan. \psi,$$

then it follows, that

$$\mu = \varphi' + \psi',$$

and thus we have an easy method of calculating the angle μ , by the tables of sines, when φ and ψ are known. It is necessary however, in the first place, to calculate $\Delta(\varphi)$ and $\Delta(\psi)$; for this purpose, take the angles α and β , such that $\sin. \alpha = c \sin. \varphi$, and $\sin. \beta = c \sin. \psi$, then will $\Delta(\varphi) = \cos. \alpha$, and $\Delta(\psi) = \cos. \beta$. By these formulas, we can find the amplitude μ of the elliptic function, which is the sum of two functions whose amplitudes are φ and ψ . We may thence also find the amplitude of a function which is the difference of two other functions; let

$$F(\varphi) - F(\psi) = F(\mu);$$

and the amplitude μ of the function, which is the difference, is determined by any one of the following equations:

$$\cos. \mu = \frac{\cos. \varphi \cos. \psi + \Delta(\varphi) \Delta(\psi) \sin. \varphi \sin. \psi}{1 - c^2 \sin.^2 \varphi \sin.^2 \psi},$$

$$\sin. \mu = \frac{\Delta(\psi) \sin. \varphi \cos. \psi - \Delta(\varphi) \sin. \psi \cos. \varphi}{1 - c^2 \sin.^2 \varphi \sin.^2 \psi},$$

$$\Delta(\mu) = \frac{\Delta(\varphi) \Delta(\psi) + c^2 \sin. \varphi \sin. \psi \cos. \varphi \cos. \psi}{1 - c^2 \sin.^2 \varphi \sin.^2 \psi},$$

$$\tan. \mu = \frac{\Delta(\psi) \tan. \varphi - \Delta(\varphi) \tan. \psi}{1 + \Delta(\varphi) \Delta(\psi) \tan. \varphi \tan. \psi},$$

in this last, if we make $\tan. \varphi' = \Delta(\psi) \tan. \varphi$, and $\tan. \psi' = \Delta(\varphi) \tan. \psi$, we shall have $\mu = \varphi' - \psi'$.

These formulas for the difference follow from those for the sum, by simply changing the sign of ψ in the latter, and keeping $\Delta(\psi)$ positive. It is besides needless to point out the analogy which subsists between the values of $\cos. \mu$, $\sin. \mu$, and those of $\cos. (\varphi \pm \psi)$, $\sin. (\varphi \pm \psi)$; they coincide entirely if c is supposed $= 0$.

17. Since we know algebraically, the amplitude of the function equal to the sum, or to the difference of two given functions, it is evident that we can find, algebraically, a multiple elliptic function of a given function, and that in general we can resolve the same problems relating to the multiplication and division of elliptic functions of the first kind, as can be resolved relating to the multiplication and division of arcs of a circle. Let φn denote the amplitude of the function which contains n times the function of which the amplitude is φ , so that we have $F(\varphi n) = n F(\varphi)$; it may be required to determine the general expression of

$\sin.$ or $\cos.$ ϕn , by means of $\sin.$ and $\cos.$ ϕ . The following are some researches relative to this subject.

The extreme cases present no difficulty. If $c = 0$, then $\phi n = n\phi$, and we have the known relation, $\cos. \phi n + \sin. \phi n \times \sqrt{-1} = (\cos. \phi + \sin. \phi \sqrt{-1})^n$. If $c = 1$, then the function $F(\phi)$ becomes $\int \frac{d\phi}{\cos. \phi}$, or $\frac{1}{2} \log. \frac{1 + \sin. \phi}{1 - \sin. \phi}$, so that, for the multiplication of functions, we have this very simple relation, $\frac{1 + \sin. \phi n}{1 - \sin. \phi n} = \left(\frac{1 + \sin. \phi}{1 - \sin. \phi} \right)^n$. The difficulty is to find the general expression of $\sin. \phi n$ when c has any value between 0 and 1.

Let us in the first place consider the most simple case, which is that of duplication, then we have

$$\cos. \phi 2 = \frac{1 - 2 \sin.^2 \phi + c^2 \sin.^4 \phi}{1 - c^2 \sin.^4 \phi},$$

$$\sin. \phi 2 = \frac{2 \Delta(\phi) \sin. \phi \cos. \phi}{1 - c^2 \sin.^4 \phi},$$

$$\tan. \frac{1}{2} \phi 2 = \Delta(\phi) \tan. \phi.$$

The last of these formulas appears convenient for trigonometrical calculation; for if we take the angles $\alpha, \alpha 2, \alpha 4$, such that $\sin. \alpha = c \sin. \phi$, $\sin. \alpha 2 = c \sin. \phi 2$, $\sin. \alpha 4 = c \sin. \phi 4$, &c. it follows that

$\tan. \frac{1}{2} \phi 2 = \cos. \alpha \tan. \phi$, $\tan. \frac{1}{2} \phi 4 = \cos. \alpha 2 \tan. \phi 2$, &c. so that the function F shall be multiplied by 2, 4, 8, 16, &c.

If it be required to divide by the same numbers; let $\phi \frac{1}{2}$ be put, by analogy, to denote the amplitude of the function which is the half of F , and the inverse of the preceding formulas give

$$\sin. \phi \frac{1}{2} = \frac{\sin. \frac{1}{2} \phi}{\sqrt{\left(\frac{1 + \Delta}{2} \right)}}.$$

Now let $\sin. \alpha = c \sin. \phi$, then $\Delta = \cos. \alpha$, and we have this very simple expression,

$$\sin. \phi \frac{1}{2} = \frac{\sin. \frac{1}{2} \phi}{\cos. \frac{1}{2} \alpha};$$

If in like manner, we make $\sin. \alpha \frac{1}{2} = c \sin. \phi \frac{1}{2}$, we shall have

$$\sin. \phi \frac{1}{4} = \frac{\sin. \frac{1}{4} \phi \frac{1}{2}}{\cos. \frac{1}{4} \alpha \frac{1}{2}},$$

and so on.

18. Let us now proceed to the multiplication and division by any number. For this purpose, let us consider the three consecutive

cutive amplitudes $\overline{\varphi n - 1}$, φn , $\overline{\varphi n + 1}$, which agreeably to the preceding notation, answer to the functions $(n - 1) F$, nF , $(n + 1) F$. The formulas for the sum and the difference of two functions apply to the equations $F(\overline{\varphi n + 1}) = F(\varphi n) + F(\varphi)$, $F(\overline{\varphi n - 1}) = F(\varphi n) - F(\varphi)$, and hence it follows, that

$$\sin. \overline{\varphi n + 1} + \sin. \overline{\varphi n - 1} = \frac{2 \Delta \cos. \varphi \sin. \varphi n}{1 - c^2 \sin.^2 \varphi \sin.^2 \varphi n};$$

$$\cos. \overline{\varphi n + 1} + \cos. \overline{\varphi n - 1} = \frac{2 \cos. \varphi \cos. \varphi n}{1 - c^2 \sin.^2 \varphi \sin.^2 \varphi n}.$$

These formulas in which Δ and φ remain always the same, while n varies, appear to be the best possible for giving the successive values of $\sin. \varphi 2$, $\sin. \varphi 3$, $\cos. \varphi 2$, $\cos. \varphi 3$, &c. Thus, putting (with a view to abridge), $2 \Delta \cos. \varphi = p$, $1 - c^2 \sin.^2 \varphi = q$, $c^2 \sin.^2 \varphi = 1 - q = r$, we have

$$\sin. \varphi 2 = \frac{p}{q} \sin. \varphi,$$

$$\sin. \varphi 3 = \frac{p^2 - r q^2}{q^2 - r p^2} \sin. \varphi,$$

$$\sin. \varphi 4 = \frac{(1 + r) p^2 - 2 q^2}{q^4 - r p^4} p q \sin. \varphi,$$

$$\sin. \varphi 5 = \frac{q^6 - 3 p^2 q^4 + (1 + 2r) p^4 q^2 - r^2 p^6}{q^6 - 3 r p^2 q^4 + r(r + 2) p^4 q^2 - r p^6} \sin. \varphi,$$

&c.

But these expressions, when completely developed, become extremely complicated, and their law is very difficult to discover.

When it is proposed to calculate trigonometrically φn by means of φ , it may be easily done as follows. The two foregoing formulas give by division

$$\frac{\sin. \overline{\varphi n + 1} + \sin. \overline{\varphi n - 1}}{\cos. \overline{\varphi n + 1} + \cos. \overline{\varphi n - 1}} = \Delta \tan. \varphi n,$$

which is reducible to this very simple formula,

$$\tan. \left(\frac{1}{2} \overline{\varphi n + 1} + \frac{1}{2} \overline{\varphi n - 1} \right) = \Delta \tan. \varphi n;$$

and hence may be deduced successively

$$\tan. \frac{1}{2} \varphi 2 = \Delta \tan. \varphi,$$

$$\tan. \left(\frac{1}{2} \varphi 3 + \frac{1}{2} \varphi \right) = \Delta \tan. \varphi 2,$$

$$\tan. \left(\frac{1}{2} \varphi 4 + \frac{1}{2} \varphi 2 \right) = \Delta \tan. \varphi 3,$$

&c.

Now Δ being always the same in these formulas, no simpler method of calculating the successive values of $\varphi 2$, $\varphi 3$, &c. by means of φ need be wished for. We may also proceed by greater intervals in the calculation of φm , m being as great as we please: for we have in like manner

$\tan.$

$\tan. (\frac{1}{2} \phi \overline{n+1} + \frac{1}{2} \phi \overline{n-1}) = \Delta i \tan. \phi n,$
 Δi being the Δ which corresponds to the amplitude ϕi .

The division of a given elliptic function into a certain number of equal parts is an algebraic problem which may be resolved by the developement of the formulas which serve for its multiplication. We have found already the formulas for its division by 2 or any power of 2; let it now be required to divide the function F into three equal parts; let ϕ_3 be the given amplitude of the function F , and ϕ that of the function which is one-third of it. We make $\sin. \phi_3 = a$, $\sin. \phi = x$, and the equation to be resolved for its trisection will be

$$a = \frac{3 - 4(1+c^2)x^2 + 6c^2x^4 - c^4x^6}{1 - 6c^2x^2 + 4c^2(1+c^2)x^4 - 3c^4x^6} \cdot x.$$

This equation is of the ninth degree; it would be of the 25th degree for the quinisection, and so on.

The equations are less complex by the half when it is proposed to divide the whole function F_1 , the amplitude of which is 90° . Suppose in general $\phi n = 90^\circ$, the formula (a') of article 15, where we may make $\mu = 90^\circ$, gives immediately this relation,

$$\tan. \phi \overline{n-1} = \frac{1}{b} \cot. \phi i;$$

hence we have successively $\tan. \phi \overline{n-1} = \frac{1}{b} \cot. \phi$,
 $\tan. \phi \overline{n-2} = \frac{1}{b} \cot. \phi_2$, $\tan. \phi \overline{n-3} = \frac{1}{b} \cot. \phi_3$, &c.
 Again, because $\phi n = 90^\circ$, we have also

$$\tan. (45^\circ + \frac{1}{2} \phi \overline{n-2}) = \Delta \tan. \phi \overline{n-1} = \frac{\Delta}{b} \cot. \phi$$

$$\tan. (\frac{1}{2} \phi \overline{n-1} + \frac{1}{2} \phi \overline{n-3}) = \Delta \tan. \phi \overline{n-2}$$

&c.

from which $\phi \overline{n-1}$, $\phi \overline{n-2}$, &c. may be determined by formulas sufficiently simple. Developing in like manner the formulas which give ϕ_2 , ϕ_3 , &c. until the two series meet, we have by their union the means of obtaining an equation which will determine ϕ ; thus when $n = 3$, that equation is directly

$$\tan. (45^\circ + \frac{1}{2} \phi) = \frac{\Delta}{b} \cot. \phi,$$

or $b \sin. \phi = \Delta (1 - \sin. \phi)$, from which, making $\sin. \phi = x$, we find

$$0 = 1 - 2x + 2c^2x^3 - c^2x^4,$$

an equation for the trisection of the function F_1 , and of which the degree is only 4 or $\frac{9-1}{2}$.

When $n = 5$ it is necessary to eliminate ϕ_3 from these two equations,

$$\tan. (45^\circ + \frac{1}{2} \phi_3) = \frac{\Delta}{b} \cot. \phi,$$

$\tan. (\frac{1}{2} \phi_3 + \frac{1}{6} \phi) = \Delta \tan. \phi_2$,
and then to substitute for $\tan. \phi_2$ its value

$\frac{2 \Delta \sin. \phi \cos. \phi}{1 - 2 \sin.^2 \phi + c^2 \sin.^4 \phi}$. We thus obtain, making $\sin. \phi = x$,

$$\frac{(1+x) \Delta}{bx} = \frac{1+2x-2c^2x^3-c^2x^4}{1-2x+2c^2x^3-c^2x^4},$$

an equation for the quinquisection of the function F_1 , and which when completely developed rises to the $\frac{25-1}{2}$ th degree.

Such are the formulas by which we may find the relation between ϕn and ϕ in order that $F(\phi n) = nF(\phi)$, n being a whole number. If it were required to find the relation between ϕ and

ψ , so that $F(\psi) = \frac{m}{n} F(\phi)$, m and n being integers, an auxiliary angle ω might be taken, such, that $nF(\psi) = F(\omega)$, and $mF(\phi) = F(\omega)$. The first condition would give an algebraical equation between the sines and cosines of the angles ψ and ω ; and the second an equation between the angles ϕ and ω , whence, eliminating ω , the required relation between ϕ and ψ would be found.

19. Hence it follows that we may always find the complete algebraic integral of the equation

$$\frac{m d\phi}{\sqrt{(1-c^2 \sin.^2 \phi)}} + \frac{n d\psi}{\sqrt{(1-c^2 \sin.^2 \psi)}} = 0,$$

m and n being whole numbers. For the integral is in the first place $mF(\phi) + nF(\psi) = \text{const.}$ and if we suppose that when $\phi = 0$, $\psi = \mu$, the constant quantity will be $nF(\mu)$, and thus we have

$$mF(\phi) + nF(\psi) = nF(\mu).$$

But an algebraic equation can be found which shall represent this transcendental equation, for if we make $F(\omega) = F(\mu) - F(\psi)$, which gives $mF(\phi) = nF(\omega)$, each of these equations may be expressed in algebraic terms, and thus, by eliminating ω , we shall have the complete algebraic integral of the proposed equation.

In general, in the following equation

$$0 = \frac{m d\phi}{\sqrt{(1-c^2 \sin.^2 \phi)}} + \frac{n d\psi}{\sqrt{(1-c^2 \sin.^2 \psi)}} + \frac{p d\omega}{\sqrt{(1-c^2 \sin.^2 \omega)}} + \&c.$$

consisting of any finite number of terms, and in which $m, n, p, \&c.$ are whole numbers positive or negative, the complete integral will be $F(\mu) = mF(\phi) + nF(\psi) + pF(\omega) + \&c.$ μ being the arbitrary constant quantity: but this transcendental equation may always be changed into an algebraic equation; for making

$F(\varphi) = mF(\psi)$, $F(\psi) = nF(\omega)$, $F(\omega) = pF(\mu)$, &c. so that $F(\mu) = F(\varphi) + F(\psi) + F(\omega) + \&c.$ each of these equations may be changed into an algebraic equation, and if we eliminate φ , ψ , ω , &c. the resulting equation will contain the arbitrary quantity μ , and will be the complete algebraic integral of the proposed equation.

If we denote the radical $\sqrt{x + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}$ by $R(x)$, and a like radical of y by $R(y)$, &c. and suppose m , n , p , &c. as before whole numbers, positive or negative, it is evident that the equation

$$0 = \frac{m dx}{R(x)} + \frac{n dy}{R(y)} + \frac{p dz}{R(z)} + \&c.$$

may always be reduced to the preceding form, and consequently will always have a complete algebraic integral.

We may suppose that z and the following variable quantities are given algebraic functions of x and y ; then the preceding equation will contain only two variable quantities, and notwithstanding the infinity of forms of which it is susceptible, it will always admit of a complete algebraic integral. It appears to me that this manner of generalizing the result of Euler relating to

the equation $\frac{dx}{R(x)} + \frac{dy}{R(y)} = 0$, has not yet been attended to. A celebrated geometer has proposed (*Mem. de Turin, Tome*

IV.) to find cases in which the equation $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$, is

integrable, X being supposed any polynomial formed from x , and Y a like polynomial from y ; but it does not appear, that his researches have gone beyond the equation of Euler; for the equation which he gives, (page 119,) as more general, is immediately reduced to it by making $v = ky$, and giving a suitable value to k . Thus it is very doubtful, whether the equation of Euler can be made more general with two terms only, but with more than two it appears that the equation may be greatly extended.

20. Since the functions F admit of being multiplied or divided by any number, we may, by means of this property, find their values by approximation. In the first place, we may suppose that φ does not exceed 90° , for we have shewn, (art. 12,) that every other case is reducible to this one. This being assumed, we may, art. 17, determine $\varphi \frac{1}{2}$, so that $F(\varphi \frac{1}{2}) = \frac{1}{2}F\varphi$, and the extent of the integral $\int \frac{d\varphi \frac{1}{2}}{\Delta(\varphi \frac{1}{2})}$ will be nearly half that of

$\int \frac{d\varphi}{\Delta(\varphi)}$, if $\epsilon \sin. \varphi$ is not very near to unity. By a second bisec-

tion,

tion, we may take $F(\phi) = \frac{1}{4}F(\phi)$, and the integral $\int \frac{d\phi}{\Delta(\phi)}$ will have only one fourth, nearly, of that extent, and so on. But when the amplitude ϕ is greatly diminished, $F(\phi)$ is reduced nearly to the arc ϕ . Therefore, whatever be the first value of ϕ , the corresponding function $F(\phi)$ is equal to the last term of the series $2\phi^{\frac{1}{2}}$, $4\phi^{\frac{1}{2}}$, $8\phi^{\frac{1}{2}}$, &c. and in most cases we may find this limit by a very short calculation.

EXAMPLE.

Let it be proposed to find the value of F when $c = \sqrt{\frac{2+\sqrt{3}}{4}}$

$= \sin. 75^\circ$, and $\tan. \phi = \sqrt{\frac{2}{\sqrt{3}}}$. By employing the formulas of art 17, and working by Gardiner's Tables, we shall find

$\phi = 47^\circ$	3'	30".91	$\alpha = 45^\circ$	0'	0".00
$\phi^{\frac{1}{2}} = 25$	36	5.64	$\alpha^{\frac{1}{2}} = 24$	40	10.94
$\phi^{\frac{1}{4}} = 13$	6	30.985	$\alpha^{\frac{1}{4}} = 12$	39	15.83
$\phi^{\frac{1}{8}} = 6$	35	40.741	$\alpha^{\frac{1}{8}} = 6$	22	8.4
$\phi^{\frac{1}{16}} = 3$	18	8.748	&c.		
&c.					

The two last values give

$$8\phi^{\frac{1}{8}} = 52^\circ 45' 25".93$$

$$16\phi^{\frac{1}{16}} = 52^\circ 50' 19".97,$$

Their difference is $4' 54".04$; and, as by the nature of these approximations, a result ought to fall short of the limit by about one-fourth of the distance of the preceding one from it, we add to the last result the third of the difference $4' 54".04$, which is $1' 38".01$, and thus get F very nearly equal to the arc $52^\circ 51'$

$57".98$, which, in parts of the radius is $\frac{\pi}{2} \times 0.5874012 = 0.9226877$.

This method may in some cases, be very tedious: we shall in the sequel give another much more expeditious.

A remark upon the motion of a simple pendulum.

21. We have shewn in the *Memoirs of the Academy* for 1786, page 637, that the motion of a simple pendulum admits of two cases: in the one it oscillates in the vertical plane, and in the other

other, it turns always in the same direction. But as the two cases lead to the same formula, we shall only consider the case of oscillations. Therefore, the force of gravity being unity, let l be the length of the pendulum, h the height due to the velocity at the lowest point, ϕ the angle which the pendulum makes with

the vertical at the end of the time t . Also, let $\frac{h}{2l} = c^2$, and

$\sin. \frac{1}{2}\phi = c \sin. \psi$, then $dt = \sqrt{l} \cdot \frac{d\psi}{\sqrt{(1 - c^2 \sin.^2 \psi)}}$, and

consequently $t = \sqrt{l} F(\psi)$.

Thus the time taken to describe any arc is an elliptic function of the first kind, and it has all the properties of these functions. Hence it follows, that the arc described in any time t being given, we can find algebraically another arc described in a time which is a multiple of t , or in general, which is commensurable with t . We can also find an arc, such, that the time of describing it shall be equal to the sum or to the difference of the times of describing several other given arcs, and this whether the arcs be adjoining to the vertical or not. These properties are new and remarkable, and it does not appear, that any one has observed that we can compare algebraically the times of portions of oscillation like as we compare circular arcs. Thus to give a very simple example, let it be proposed to divide into two equal parts the time of a semi-oscillation. Let ϕ be the arc reckoned from the vertical to the point which answers to the middle of the semi-oscillation; then I say that the chord of ϕ shall be to the chord of $\frac{1}{2}a$ as $\sqrt{2}$ is to 1. For since $F(\psi) = \frac{1}{2}F(90^\circ)$, the

formula of art. 17 gives $\sin.^2 \psi = \frac{1}{1 + \sqrt{(1 - c^2)}}$, and there-

fore $\sin.^2 \frac{1}{2}\phi = \frac{c^2}{1 + \sqrt{(1 - c^2)}}$; but it is evident that $c = \sin.$

$\frac{1}{2}a$; therefore $\sin.^2 \frac{1}{2}\phi = 1 - \cos. \frac{1}{2}a = 2 \sin.^2 \frac{1}{4}a$, and hence $2 \sin. \frac{1}{2}\phi : 2 \sin. \frac{1}{4}a :: \sqrt{2} : 1$.

Comparison of Elliptic Functions of the second kind.

22. Let us consider two elliptic functions of the second species, which only differ in respect of their amplitudes ϕ and ψ , so that we have

$$G(\phi) = \int \frac{(A + B \sin.^2 \phi) d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}}, \quad G(\psi) = \int \frac{(A + B \sin.^2 \psi) d\psi}{\sqrt{(1 - c^2 \sin.^2 \psi)}}.$$

Let us suppose that the angles ϕ and ψ have to each other the relation expressed by the equation $F(\phi) + F(\psi) = F(\mu)$, μ being

being a constant quantity, which gives

$$\frac{d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}} + \frac{d\psi}{\sqrt{(1 - c^2 \sin.^2 \psi)}} = 0;$$

hence there results $G(\phi) + G(\psi) = \int \frac{B(\sin.^2 \phi - \sin.^2 \psi) d\phi}{\sqrt{(1 - c^2 \sin.^2 \phi)}}$
 $+ \text{const.}$ But when $\phi = 0$, then $\psi = \mu$; thus, supposing
the integral taken from $\phi = 0$, we have

$$G(\phi) + G(\psi) = G(\mu) + \int B(\sin.^2 \phi - \sin.^2 \psi) \frac{d\phi}{\Delta(\phi)}.$$

But the equation $F(\phi) + F(\psi) = F(\mu)$, is in effect, the same
as equation (a') of article 15, and from which we find, after
some reductions,

$$d\phi \Delta(\phi) + d\psi \Delta(\psi) = c^2 \sin. \mu d(\sin. \phi \sin. \psi).$$

Besides, we have $\frac{d\phi}{\Delta(\phi)} + \frac{d\psi}{\Delta(\psi)} = 0$, therefore

$(\sin.^2 \phi - \sin.^2 \psi) \frac{d\phi}{\Delta} = -\sin. \mu d(\sin. \phi \sin. \psi)$. Hence, upon
the whole,

$$G(\phi) + G(\psi) - G(\mu) = -B \sin. \mu \sin. \phi \sin. \psi.$$

Thus it appears, that the same relation between the angles ϕ ,
 ψ , μ , which gives $F(\phi) + F(\psi) - F(\mu) = 0$, gives $G(\phi) +$
 $G(\psi) - G(\mu)$, if not equal to 0, at least equal to a very simple
algebraic quantity. This equation is the source of many conse-
quences analogous to those which have been deduced from the
function F . If we denote as before, by ϕ_2 , ϕ_3 , ϕ_4 , &c. ampli-
tudes such, that $F(\phi_2) = 2F(\phi)$, $F(\phi_3) = 3F(\phi)$, &c. we
have, in consequence of this equation,

$$G(\phi_2) - 2G(\phi) = B \sin. \phi \sin. \phi \sin. \phi_2,$$

$$G(\phi_3) - 3G(\phi) = B \sin. \phi (\sin. \phi \sin. \phi_2 + \sin. \phi_2 \sin. \phi_3)$$

$$G(\phi_4) - 4G(\phi) = B \sin. \phi \left\{ \sin. \phi \sin. \phi_2 + \sin. \phi_2 \sin. \phi_3 \right\},$$

&c.

Therefore the same relation between ϕ_n and ϕ , which gives
 $F(\phi_n) = nF(\phi)$, gives $G(\phi_n) - nG(\phi)$ equal to an algebraic
quantity.

23. It appears, without entering more into detail, that if m ,
 n , p , are whole numbers, positive or negative, we may have

$$mG(\phi) + nG(\psi) + pG(\omega) + \&c.$$

equal to an algebraic quantity. That this may be the case, it is
necessary to determine the relation between the angles ϕ , ψ , ω ,
&c. which gives

$$0 = mF(\phi) + nF(\psi) + pF(\omega) + \&c.$$

We observe that the functions $F(\phi)$, $G(\phi)$, have in general,
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the same sign as ϕ ; when ϕ changes its sign, they also change their signs, retaining the same values. This being premised, we may satisfy the equation

$$0 = mF(\phi) + nF(\psi) + pF(\omega) + \&c.$$

in many different ways, for we have it in our power to change the sign of each coefficient, provided that the sign of the corresponding amplitude be changed at the same time. But we may limit the functions $F(\phi)$, $F(\psi)$, $G(\phi)$, &c. to positive values, and in this case there is only one relation between the angles ϕ , ψ , ω , &c. which gives $0 = mF(\phi) + nF(\psi) + pF(\omega) + \&c.$ then it appears that the coefficients m , n , p , &c. cannot be all positive.

Whatever has been demonstrated of the function G in general, applies to elliptic arcs in particular, and is easily extended to hyperbolic arcs. The results agree with those we have found in the volume already quoted, but the connection that subsists between these properties, is here better perceived by the relations between elliptic functions of the second species, and those of the first. We shall see immediately, that elliptic functions of the third kind, have similar properties, with this gradation, that the difference which is $= 0$ in the first kind of functions, and which is an algebraic quantity in the second kind, is a transcendental quantity in the third, but may always be determined by arcs of a circle and logarithms.

Comparison of Elliptic Functions of the third kind.

24. The general formula of these functions, is

$$H(\phi) = \int \frac{A + B \sin^2 \phi}{1 + n \sin^2 \phi} \cdot \frac{d\phi}{\Delta(\phi)};$$

If we consider a like function of ψ , and suppose that between ϕ , ψ , and the constant arc μ , the equation $F(\phi) + F(\psi) -$

$F(\mu) = 0$, takes place, then because $\frac{d\phi}{\Delta(\phi)} + \frac{d\psi}{\Delta(\psi)} = 0$,

$$\begin{aligned} H(\phi) + H(\psi) - H(\mu) &= \int \left\{ \frac{A + B \sin^2 \phi}{1 + n \sin^2 \phi} \cdot \frac{d\phi}{\Delta(\phi)} + \frac{A + B \sin^2 \psi}{1 + n \sin^2 \psi} \cdot \frac{d\psi}{\Delta(\psi)} \right\} \\ &= \int \frac{(B - An) (\sin^2 \phi - \sin^2 \psi)}{1 + n (\sin^2 \phi + \sin^2 \psi) + n^2 \sin^2 \phi \sin^2 \psi} \cdot \frac{d\phi}{\Delta(\phi)}. \end{aligned}$$

Let $\sin^2 \phi + \sin^2 \psi = p$, $\sin \phi \sin \psi = q$; we have already found, (art. 22.), $(\sin^2 \phi - \sin^2 \psi) \frac{d\phi}{\Delta} = -\sin \mu d(\sin \phi \sin \psi)$, thus we have

$$H(\phi) + H(\psi) - H(\mu) = \int \frac{(An - B) \sin \mu dq}{1 + np + n^2 q^2};$$

But

But the algebraic equation between ϕ , ψ , μ is $\cos. \phi \cos. \psi = \Delta(\mu) \sin. \phi \sin. \psi + \cos. \mu$; by squaring this equation, and making the necessary substitutions, we thence deduce $p = \sin.^2 \mu - 2q \Delta(\mu) \cos. \mu + q^2 \sin.^2 \mu$, which value being put in the preceding formula, gives

$$H(\phi) + H(\psi) - H(\mu) = \int \frac{(An - B) \sin. \mu d\phi}{1 + n \sin.^2 \mu - 2nq \Delta(\mu) \cos. \mu + q^2 (n^2 + n \cos.^2 \mu)}$$

the integral in this expression ought to be taken from $q = 0$.

From this equation, and all those that can be formed in the same manner between three functions, we conclude that if $m, n, p, \&c.$ are whole numbers, positive or negative, we can always express

$$m'H(\phi) + n'H(\psi) + p'H(\omega) + \&c.$$

by arcs of a circle or logarithms. To effect this, we must establish such a relation among the amplitudes $\phi, \psi, \omega, \&c.$ that

$$mF(\phi) + nF(\psi) + pF(\omega) + \&c. = 0.$$

This result is subject to no exception; it even takes place when the coefficients A, B, n , are imaginary. But these properties may be considered under a more general point of view.

25. Let P , or $P(\phi)$, be a rational even function of $\sin. \phi$, and let

$$Z(\phi) = \int \frac{P(\phi) d\phi}{\sqrt{1 - c^2 \sin.^2 \phi}};$$

let $Z(\psi)$ be a function similar to $Z(\phi)$, and let us suppose that we have always the equation $\cos. \phi \cos. \psi = \Delta(\mu) \sin. \phi \sin. \psi$

+ $\cos. \mu$, which, considering μ as constant, gives $\frac{d\phi}{\Delta(\phi)} +$

$\frac{d\psi}{\Delta(\psi)} = 0$, we have, therefore

$$Z(\phi) + Z(\psi) - Z(\mu) = \int \{P(\phi) - P(\psi)\} \frac{d\phi}{\Delta(\phi)}.$$

Make as before, $\sin.^2 \phi + \sin.^2 \psi = p$, $\sin. \phi \sin. \psi = q$, and we get

$$\sin.^2 \phi = \frac{1}{2}p + \frac{1}{2}\sqrt{(p^2 - 4q^2)}$$

$$\sin.^2 \psi = \frac{1}{2}p - \frac{1}{2}\sqrt{(p^2 - 4q^2)}.$$

Let us now substitute the value of $\sin.^2 \phi$ in P , the result will be of the form

$$P(\phi) = M + N\sqrt{(p^2 - 4q^2)},$$

M and N being rational functions of p and q . It is evident, that we have, in like manner,

$$P(\psi) = M - N\sqrt{(p^2 - 4q^2)};$$

therefore $P(\varphi) - P(\psi) = 2N \sqrt{p^2 - 4q^2} = 2N (\sin^2 \varphi - \sin^2 \psi)$, and thus the integral $\int \{P(\varphi) - P(\psi)\} \frac{d\varphi}{\Delta(\varphi)} = \int 2N (\sin^2 \varphi - \sin^2 \psi) \frac{d\varphi}{\Delta(\varphi)} = -2 \sin \mu \int N dq$, and finally

$$Z(\varphi) + Z(\psi) - Z(\mu) = -2 \sin \mu \int N dq.$$

In this formula, N is a rational function of p and q ; if we substitute in it the value of p , found in art. 24, N will be a rational function of q only, and thus the value of $Z(\varphi) + Z(\psi) - Z(\mu)$ may always be determined by arcs of a circle and logarithms.

In general, if $m, n, p, \&c.$ denote whole numbers, positive or negative, and we establish among the angles φ, ψ, ω , the relation $0 = mF(\varphi) + nF(\psi) + pF(\omega)$, the same relation will give

$$m'Z(\varphi) + n'Z(\psi) + p'Z(\omega) + \&c.$$

equal to a quantity determinable by arcs of a circle and logarithms. The same property also holds true even when P contains odd powers of $\sin \varphi$, for the part of Z affected by the uneven powers, is integrable by the common rules.

The very same is true of the function

$$Z(x) = \int \frac{P dx}{\sqrt{(x + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4)}},$$

P being a rational function of x , and we can always find an algebraic equation between $x, y, z, \&c.$ by means of which

$$m'Z(x) + n'Z(y) + p'Z(z) + \&c.$$

is determinable by arcs of a circle, and logarithms.

Method of Approximation applied to Elliptic Functions of the first kind.

26. The method we are to employ is the same as that we have used in the volume of 1786, already quoted, in order to reduce the rectification of a given ellipse to that of two other ellipses as little different from a circle as we please. The spirit of this method consists in reducing the integral of a differential affected with the radical $\sqrt{(1 - c^2 \sin^2 \varphi)}$ to that of a like differential in which the modulus c is less than any given quantity.

In the case of the rectification of the ellipse the modulus c , represents the eccentricity, and its complement $\sqrt{(1 - c^2)}$, or b is the lesser axis. We have shewn that, supposing always the semi-transverse axis $= 1$, all the ellipses which can be thus rectified by means of two of them, and which compose the same series

series or family, have their excentricities $c, c', c'',$ &c. determined by the following law,

$$c' = \frac{2\sqrt{c}}{1+c}, c'' = \frac{2\sqrt{c'}}{1+c'}, c''' = \frac{2\sqrt{c''}}{1+c''}, \&c.$$

In this direction, the excentricities increase continually, and approach rapidly to unity; they decrease with the same rapidity in the contrary direction. We shall denote the same series continued backwards by $c, c^o, c^{\infty},$ &c. so that the complete series of excentricities, which has on the one hand, 0 for a limit, and on the other unity, is

$$0 \dots c^{\infty}, c^o, c, c', c'', c''', \dots 1,$$

we have, therefore similarly,

$$c = \frac{2\sqrt{c^o}}{1+c^o}, c^o = \frac{2\sqrt{c^{\infty}}}{1+c^{\infty}}, \&c.$$

But to determine c^o from c , we employ the formula $c^2 =$

$$\frac{1 - \sqrt{1-c^2}}{1 + \sqrt{1-c^2}}, \text{ or } c^o = \frac{1-b}{1+b}, \text{ and thus we have}$$

$$c = \frac{1-b}{1+b}, c^o = \frac{1-b^o}{1+b^o}, c^{\infty} = \frac{1-b^{\infty}}{1+b^{\infty}}, \&c.$$

It is by this law, we determine the decreasing *moduli* $c^o, c^{\infty},$ &c. of which we shall make frequent use in the following investigations. We may also employ the formulas

$$b^o = \frac{2\sqrt{b}}{1+b}, b^{\infty} = \frac{2\sqrt{b^o}}{1+b^o}, \&c.$$

Lastly, we shall afterwards shew how they may be found by the tables of sines.

27. This being premised, let us consider the integral formula,

$$F = \int \frac{d\varphi}{\sqrt{(1-c^2 \sin.^2 \varphi)}}, \text{ and let there be taken a new variable arc } \varphi^o, \text{ such that}$$

$$2 \sin. \varphi^2 = 1 + c^o \sin.^2 \varphi^o - \Delta^o \cos. \varphi^o;$$

we have shewn how c^o is to be found from c ; as to Δ^o , it is put by analogy for $\sqrt{(1-c^o \sin.^2 \varphi^o)}$. From this supposition, we get

$$2 \cos.^2 \varphi = 1 - c^o \sin.^2 \varphi^o + \Delta^o \cos. \varphi^o,$$

$$\Delta = \frac{c^o \cos. \varphi^o + \Delta^o}{1 + c^o},$$

$$2 \sin. \varphi \cos. \varphi = \sin. \varphi^o (c^o \cos. \varphi^o + \Delta^o);$$

Again, the differential of $2 \sin.^2 \varphi$ being taken, gives

$$4 \sin. \varphi \cos. \varphi d\varphi = \frac{d\varphi^o \sin. \varphi^o}{\Delta^o} (c^o \cos. \varphi^o + \Delta^o)^2;$$

and dividing this equation by the last, we have

$$2d\varphi = \frac{d\varphi^0}{\Delta^0} (c^0 \cos. \varphi^0 + \Delta^0).$$

Therefore, at last we find

$$\frac{d\varphi}{\Delta} = \frac{1+c^0}{2} \cdot \frac{d\varphi^0}{\Delta^0},$$

and integrating, $\int \frac{d\varphi}{\Delta} = \frac{1+c^0}{2} \int \frac{d\varphi^0}{\Delta^0}$, or simply $F = \frac{1+c^0}{2} F^0$, where F^0 denotes the new elliptic function $\int \frac{d\varphi^0}{\Delta^0}$.

It is evident, that by this substitution, which gives a very simple result, the determination of the function F , is reduced to that of F^0 ; which is more easily valued, because c^0 is less than c ; and it is worthy of remark, that we have only had recourse to a single new function, and not to two, as in the case of elliptic arcs, so that the result in the present case is much more simple. It remains to determine φ^0 by means of φ , but the assumed relation between these two angles, gives

$$\sin. \varphi^0 = \frac{(1+b) \sin. \varphi \cos. \varphi}{\Delta},$$

$$\cos. \varphi^0 = \frac{1 - (1+b) \sin.^2 \varphi}{\Delta},$$

$$\tan. \varphi^0 = \frac{(1+b) \tan. \varphi}{1 - b \tan.^2 \varphi}.$$

This last result may be put under the very simple form,

$$\tan. (\varphi^0 - \varphi) = b \tan. \varphi.$$

Let us therefore suppose, that after having determined the decreasing *moduli* $c^0, c^{00}, \&c.$ as also their complements $b^0, b^{00}, \&c.$ as has been explained in the preceding article, we calculate successively the amplitudes $\varphi^0, \varphi^{00}, \varphi^{000}, \&c.$ by the formulas

$$\begin{aligned} \tan. (\varphi^0 - \varphi) &= b \tan. \varphi, \\ \tan. (\varphi^{00} - \varphi^0) &= b^0 \tan. \varphi^0, \\ \tan. (\varphi^{000} - \varphi^{00}) &= b^{00} \tan. \varphi^{00}, \\ &\&c. \end{aligned}$$

which may also be done by the series

$$\begin{aligned} \varphi^0 &= 2\varphi - c^0 \sin. 2\varphi + \frac{1}{2}c^{02} \sin. 4\varphi - \frac{1}{2}c^{03} \sin. 6\varphi + \&c. \\ \varphi^{00} &= 2\varphi^0 - c^{00} \sin. 2\varphi^0 + \frac{1}{2}c^{002} \sin. 4\varphi^0 - \frac{1}{2}c^{003} \sin. 6\varphi^0 + \&c. \end{aligned}$$

Then

Then the integral $\int \frac{d\varphi}{\Delta}$, or the function F will be expressed thus,

$$F = \frac{1 + c^0}{2} \cdot F^0,$$

$$F = \frac{1 + c^0}{2} \cdot \frac{1 + c^{\infty}}{2} \cdot F^{\infty},$$

$$F = \frac{1 + c^0}{2} \cdot \frac{1 + c^{\infty}}{2} \cdot \frac{1 + c^{\infty\infty}}{2} \cdot F^{\infty\infty},$$

&c.

But when c is become very small, we have $\Delta = 1$, and $\int \frac{d\varphi}{\Delta} = \varphi$. Therefore, let Φ be the limit of the angles φ , $\frac{\varphi^0}{2}$, $\frac{\varphi^{\infty}}{4}$, $\frac{\varphi^{\infty\infty}}{8}$, &c. to which limit they approach extremely near after a few terms, and we have

$$F = \Phi (1 + c^0) (1 + c^{\infty}) (1 + c^{\infty\infty}), \text{ \&c.}$$

When $\varphi = 90^\circ = \frac{1}{2}\pi$, the limit Φ is also $\frac{1}{2}\pi$, so that the value of F , which is then F_1 , becomes

$$F_1 = \frac{\pi}{2} (1 + c^0) (1 + c^{\infty}) (1 + c^{\infty\infty}), \text{ \&c.}$$

The constant product $(1 + c^0) (1 + c^{\infty})$ &c. which we shall call α , may be also represented by the expression

$$\alpha = \frac{2\sqrt{c^0}}{c} \cdot \frac{2\sqrt{c^{\infty}}}{c^0} \cdot \frac{2\sqrt{c^{\infty\infty}}}{c^{\infty}}, \text{ \&c.}$$

and under this form, it may easily be calculated by logarithms. The value of α being known, we have, in general $F = \alpha\Phi$ and $F_1 = \alpha \cdot \frac{1}{2}\pi$.

28. The quickness of these approximations is best perceived in their applications. To render them as easy as possible, let it be observed, that if we make $c = \sin. \mu$, we have $b = \cos. \mu$, and $c^0 = \tan.^2 \frac{1}{2}\mu$, from which it appears that c^0 is easily deduced from c by the table of sines. Making in like manner $c^0 = \tan.^2 \frac{1}{2}\mu = \sin. \mu^0$, which gives a new angle μ^0 , we have $c^{\infty} = \tan.^2 \frac{1}{2}\mu^0$, and so on. When we at last come to a very small c , in order to find the next one, which I denote by c^0 , we may, to avoid small angles, employ the formula

$$c^0 = \frac{1}{4}c^2 + \frac{1.3}{4.6} c^4 + \frac{1.3.5}{4.6.8} c^6 + c^8 \text{ \&c.}$$

of which the first term, or at most the first two, will be sufficient.

As

As to the calculation of the angles φ° , $\varphi^{\circ\circ}$, &c. we have nothing to add to the simplicity of the formula $\tan. (\varphi^\circ - \varphi) = b \tan. \varphi$, which is well adapted to trigonometrical calculation; we shall only observe, that the angle $\varphi^\circ - \varphi$, thus determined by its tangent, is almost equal to the angle φ , when c becomes very small, for we have, without any ambiguity,

$$\varphi^\circ = 2\varphi - c^\circ \sin. 2\varphi + \frac{1}{2}c^{\circ 2} \sin. 4\varphi - \frac{1}{3}c^{\circ 3} \sin. 6\varphi + \&c.$$

It is therefore necessary to take for the value of $\varphi^\circ - \varphi$, not always the smallest angle which is found from the table of sines, but that which approaches most to φ , and which may be several circumferences. An example will shew, that on this point there is neither difficulty nor ambiguity.

EXAMPLE.

Required the value of the function F , when $c = \frac{1}{2}\sqrt{(2 + \sqrt{3})}$ and $\tan. \varphi = \sqrt{\frac{2}{3}}$. We purposely choose a case unfavourable to calculation, because the value of c differs but little from unity.

The following table shews how the values of c° , $c^{\circ\circ}$, b° , $b^{\circ\circ}$, &c. are calculated, according to what has been explained.

<i>Values of c and b.</i>				<i>Their logarithms.</i>
c	=	$\sin. 75^\circ 0'$	$0''$	 9.9849438.
b	=	$\cos. 75^\circ 0'$	$0''$	 9.4129962.
c°	=	$\left\{ \begin{array}{l} \tan. 37^\circ 30' \\ \sin. 36^\circ 47'47'' \end{array} \right\}$	$0''$	 9.7699610.
b°	=	$\cos. 36^\circ 47'47''$		 9.9075648.
$c^{\circ\circ}$	=	$\left\{ \begin{array}{l} \tan. 18^\circ 23'35'' \\ \sin. 6^\circ 59'36'' \end{array} \right\}$		 9.0250880.
$b^{\circ\circ}$	=	$\cos. 6^\circ 59'36''$		 9.9975454.
$c^{\circ\circ\circ}$	=	$\left\{ \begin{array}{l} \tan. 9^\circ 11'54'' \\ \sin. 3^\circ 29'42'' \end{array} \right\}$		 7.4511672.
$b^{\circ\circ\circ}$	=	$\cos. 3^\circ 29'42''$		 9.9999983.
$c^{\circ\circ\circ\circ}$	=	$\frac{1}{2}(c^{\circ\circ\circ})^2$		 4.3002761.
$b^{\circ\circ\circ\circ}$	=			 0.0000000.

These logarithms give readily the product $\alpha = \frac{2\sqrt{c^\circ}}{c} \cdot \frac{2\sqrt{c^{\circ\circ}}}{c^{\circ\circ}}$,

&c. for which four factors are sufficient, and we have $\log. \alpha = 0.2460561$, or $\alpha = 1.7622037$. Hence we find directly $F_1 =$

$$\frac{\pi}{2} \cdot \alpha = 2.7680632. \text{ We next find}$$

ϕ	$=$	47°	3'	30".96.
ϕ^o	$=$	62	36	3'12.
ϕ^{oo}	$=$	119	55	47".69.
ϕ^{ooo}	$=$	249	0	0".18.
ϕ^{ooo^o}	$=$	489	0	0".00.
&c.				

The other values of ϕ increase in a double ratio, from which it follows that the limit of the angles ϕ , $\frac{\phi^o}{2}$, $\frac{\phi^{oo}}{4}$, &c. is $= 30^\circ$ or $\frac{\pi}{6}$, and thus we have $F = \frac{\pi}{6}\alpha = 0.9226877$, as it art. 20.

Let us remark, that since the limit $\phi = 30^\circ = \frac{1}{2}90^\circ$, we have $F = \frac{1}{3}F_1$; this equality is at least approximate, since such is the value found by the calculation; but it is easy to be assured that it is accurately so; for it has appeared (art. 18.), that the equation to be resolved to make $F(\phi) = \frac{1}{3}F_1$ is

$$0 = 1 - 2 \sin. \phi + 2c^2 \sin.^3 \phi - c^2 \sin.^4 \phi;$$

but in the present case where $c^2 = \frac{2 + \sqrt{3}}{4}$ we shall find $\sin. \phi = \sqrt{3} - 1$, or $\tan. \phi = \sqrt{\left(\frac{2}{\sqrt{3}}\right)}$; therefore the function F is accurately one-third of $F(90^\circ)$ or F_1 .

29. The value of ϕ being given, it appears that the function F may be easily found to the necessary degree of accuracy. On the other hand it may be useful to determine the amplitude ϕ , when the value of the function F is supposed known. Let α always denote the product $(1 + c^o)(1 + c^{oo})$ &c. and ϕ the limit of the angles ϕ , $\frac{\phi^o}{2}$, $\frac{\phi^{oo}}{4}$, &c. Seeing we have $F = \alpha\phi$,

it follows that $\phi = \frac{F}{\alpha}$; thus the limit ϕ is known. Let us

suppose that the values of c^o , c^{oo} , &c. have been calculated to a term so small as to admit of being neglected, and for example, let this term be c^{oooo} , or, for the sake of brevity, c^{os} . Since, therefore, $\phi^{os} = 2\phi^{oo} - c^{os} \sin. 2\phi^{oo} + \&c.$ we have $\phi^{os} = 2\phi^{oo}$, very near, and much more $\phi^{ooo} = 2\phi^{os}$ &c. Therefore the limit ϕ is equal to $\frac{1}{4}\phi^{os}$, that is we have $\phi^{ooo} = 16\phi$. This last ϕ being known, we return to those which precede it by the successive equations

$$\begin{aligned} \sin. (2\varphi^{1000} - \varphi^{1000}) &= c^{1000} \sin. \varphi^{1000}, \\ \sin. (2\varphi^{100} - \varphi^{100}) &= c^{100} \sin. \varphi^{100}, \\ \sin. (2\varphi^{10} - \varphi^{10}) &= c^{10} \sin. \varphi^{10}, \\ \sin. (2\varphi - \varphi) &= c \sin. \varphi. \end{aligned}$$

These have all the same form, and the last of them is deducible from the equation $\tan. (\varphi - \varphi) = b \tan. \varphi$.

This is particularly applicable to the resolution of the equation $F(\varphi n) = nF(\varphi)$ whatever be the value of n . The angle φ being given we can find $F(\varphi)$ and $nF(\varphi)$; therefore $F(\varphi n)$ will be known, and thence φn may be found as has been explained. This method in no case requires many operations, whereas if n were somewhat great, or only fractional, the methods we have formerly given to determine φn may become very tedious, or even impracticable. Thus the present method is almost always preferable.

(To be continued in the next Volume.)

End of the Third Part of the Second Volume.







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